

THE NICHOLS ALGEBRA OF A SEMISIMPLE YETTER-DRINFELD MODULE

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ABSTRACT. We study the Nichols algebra of a semisimple Yetter-Drinfeld module and introduce new invariants such as real roots. The crucial ingredient is a “reflection” in the class of such Nichols algebras. We conclude the classifications of finite-dimensional pointed Hopf algebras over \mathbb{S}_3 , and of finite-dimensional Nichols algebras over \mathbb{S}_4 .

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INTRODUCTION

1. Although semisimple complex Lie algebras \mathfrak{g} cannot be deformed there are highly interesting q -deformations $U_q(\mathfrak{g})$ of their enveloping algebras as Hopf algebras with generic q introduced by Drinfeld and Jimbo around 1985.

As an algebra, $U_q(\mathfrak{g})$ is generated by elements $E_i, F_i, K_i^{\pm}, 1 \leq i \leq n$. The Hopf algebra $U_q(\mathfrak{g})$ is determined by the $+$ -part $U_q^+(\mathfrak{g}) = k\langle E_1, \dots, E_n \rangle$ since $U_q(\mathfrak{g})$ is essentially a Drinfeld double of $U_q^+(\mathfrak{g})$. The algebra $U_q^+(\mathfrak{g})$ has a very easy and beautiful description as the Nichols algebra (or quantum symmetric algebra) $\mathcal{B}(W)$ of a finite-dimensional vector space

$$(0.1) \quad W = \mathbb{C}E_1 \oplus \dots \oplus \mathbb{C}E_n$$

together with a grading and an action of a free abelian group G with basis K_1, \dots, K_n . Each E_i has degree K_i and the action of G is given by

$$(0.2) \quad K_i E_j K_i^{-1} = q^{d_i a_{ij}} \text{ for all } i, j.$$

Here $(d_i a_{ij})$ is the symmetrized Cartan matrix. Thus W has the structure of a Yetter-Drinfeld module over the group algebra $k[G]$, and the vector spaces $\mathbb{C}E_i$ are one-dimensional Yetter-Drinfeld submodules. See Section 1 for the definition of Yetter-Drinfeld modules.

In the same way Nichols algebras also determine the small quantum groups $u_q(\mathfrak{g})$, q a root of unity, introduced by Lusztig, and the generalizations of the quantum groups $U_q(\mathfrak{g})$ to Kac-Moody Lie algebras, see [L93, R95, R98, S96].

2. Nichols algebras appeared in the work of Nichols [N78]. They are defined for Yetter-Drinfeld modules W over any Hopf algebra H (with bijective antipode) instead of the group algebra $H = k[G]$. The category of Yetter-Drinfeld modules over H is braided, and the Nichols algebra can be defined by the following universal property: The tensor algebra $T(W)$ is a braided Hopf algebra where the elements of W are primitive. Then

$$(0.3) \quad \mathcal{B}(W) = T(W)/I_W,$$

where I_W is the largest coideal of $T(W)$ spanned by elements of \mathbb{N} -degree ≥ 2 .

The smash product $\mathcal{B}(W) \# H$ (called bosonization) is a Hopf algebra in the usual sense, and to understand Nichols algebras of Yetter-Drinfeld modules in general is of fundamental importance for the general theory of Hopf algebras. Nichols algebras form a crucial part of the \mathbb{N} -graded Hopf algebra associated to the coradical filtration of a Hopf algebra whose coradical is a Hopf subalgebra [AS98]. An important class of such Hopf algebras are pointed Hopf algebras, that is Hopf algebras where the coradical is a group algebra (or equivalently, where all the simple comodules are one-dimensional). The quantum groups $U_q(\mathfrak{g})$ and all their variants are pointed.

The definition of the Nichols algebra is easy. The inherent conceptual difficulty of understanding Nichols algebras is their very indirect definition

by a universal property. In general there is no method to actually determine the Nichols algebra $\mathcal{B}(W)$ for a given W , for example to calculate the dimensions of the \mathbb{N} -homogeneous components of $\mathcal{B}(W)$ or to compute the defining relations, that is to compute generators of the unknown ideal I_W .

The relations of the Nichols algebra of the Yetter-Drinfeld module (0.1) are the quantized Serre relations, see [L93, 33.1.5] for a proof of this deep result.

3. During the last few years several classification results for Hopf algebras were obtained based on the theory of Nichols algebras and following the procedure proposed in [AS98]. This program has been particularly successful for finite-dimensional pointed Hopf algebras with abelian group of group-like elements [AS05].

Let H be a finite-dimensional cosemisimple complex Hopf algebra and let us consider a finite-dimensional Hopf algebra A such that its coradical A_0 is a Hopf algebra isomorphic to H . To solve the problem of classifying all such Hopf algebras A , we have to address two fundamental questions: Given a Yetter-Drinfeld module W over H ,

- (a) decide when $\dim \mathcal{B}(W) < \infty$, and
- (b) describe a suitable set of defining relations of $\mathcal{B}(W)$.

Now, since H is cosemisimple, it is also semisimple [LR88]; then the category ${}^H_H\mathcal{YD}$ of Yetter-Drinfeld modules over H is semisimple [Ra93]. Therefore we just need to consider the questions (a) and (b) in the following cases:

- (i) when W is an irreducible Yetter-Drinfeld module, and
- (ii) when $W = V_1 \oplus \cdots \oplus V_\theta$ is a direct sum of irreducible Yetter-Drinfeld modules, under the assumption that the answers to questions (a) and (b) are known for V_1, \dots, V_θ .

As applications of the main results of the present paper we obtain new information in case (ii) when not all the V_i are one-dimensional.

If $V = \mathbb{C}v$ is a one-dimensional Yetter-Drinfeld submodule over H , then it determines a group-like element $g \in G(H)$ and a character $\chi \in \text{Alg}(H, \mathbb{C})$ defining the coaction and the action of H . Let $q = \chi(g)$. The Nichols algebra of V is easy to determine: it is either the polynomial algebra $\mathbb{C}[v]$, when $q = 1$ or is not a root of 1, or else it is the truncated polynomial algebra $\mathbb{C}[v]/(v^N)$, when q is a root of 1 of order $N > 1$. In other words, questions (a) and (b) have completely satisfactory answers in this case.

Assume next that $W = \mathbb{C}v_1 \oplus \cdots \oplus \mathbb{C}v_\theta$ is a direct sum of *one-dimensional* Yetter-Drinfeld submodules over H . Let $g_i \in G(H)$ and $\chi_i \in \text{Alg}(H, \mathbb{C})$ determined by the submodule $\mathbb{C}v_i$ as above. Let $q_{ij} = \chi_j(g_i)$, $1 \leq i, j \leq \theta$. Note that in the classical situation of the Yetter-Drinfeld module (0.1) for q a root of unity, $q_{ij} = q^{d_i a_{ij}}$. The Nichols algebra of W can be viewed as a “gluing” of the various Nichols subalgebras $\mathcal{B}(\mathbb{C}v_i)$ along the generalized Dynkin diagram with vertices $1, \dots, \theta$; there is a line joining the vertices i

and j if $q_{ij}q_{ji} \neq 1$, and then the line is labelled by the scalar $q_{ij}q_{ji}$, resembling the classical Killing-Cartan classification of semisimple Lie algebras.

Assume moreover that W is of Cartan type, that is, there exist $a_{ij} \in \mathbb{Z}$ such that $q_{ij}q_{ji} = q_{ii}^{a_{ij}}$ for any $i \neq j$; the classical example of Cartan type is $q_{ij} = q^{d_i a_{ij}}$ for all i, j where $(d_i a_{ij})$ is the symmetrized Cartan matrix. Then $A = (a_{ij})$ is a generalized Cartan matrix, and we have complete answers to questions (a) and (b) above:

- (a) $\dim \mathcal{B}(W) < \infty$ if and only if A is of finite type.
- (b) If $\dim \mathcal{B}(W) < \infty$, then the ideal of relations of $\mathcal{B}(W)$ is generated by the quantum Serre relations and appropriate powers of the root vectors.

These results were proved in [AS00] under some restrictions on the orders of the q_{ij} 's by reduction to the theory of quantum groups. Part (a) was shown without any restriction in [H06].

The classification of the matrices (q_{ij}) whose corresponding Nichols algebras are finite-dimensional is given in [H09]. In general, one Cartan matrix is not sufficient to describe the Nichols algebra, a family of generalized Cartan matrices is needed. The main instrument to control them is the Weyl groupoid – introduced already in [H06]. As for the defining relations, these are not yet known, except in the standard case [Ag09] when all the Cartan matrices are the same, and in the general case of rank two, that is $\theta = 2$ [H07]. Their precise description is more delicate than (b) above.

4. Let now $W = V_1 \oplus \cdots \oplus V_\theta$ be an arbitrary direct sum of irreducible Yetter-Drinfeld modules. In analogy with the situation of Cartan type, it was proposed to consider the V_i 's as “fat points” of a generalized Dynkin diagram (or some kind of generalized Cartan matrix) and, assuming the knowledge of the Nichols algebras $\mathcal{B}(V_i)$, to describe the Nichols algebra $\mathcal{B}(W)$ as a “gluing” of the various Nichols subalgebras $\mathcal{B}(V_i)$ along it [A02, p. 41]. Because of [H09], it is clear that just one generalized Cartan matrix would not be enough, and that we would need to attach to our W a collection of generalized Cartan matrices. This is what we do in the present paper.

5. Let us now proceed with a detailed description of our results which in fact hold in a much more general context. Let \mathbb{k} be an arbitrary field and let H be any Hopf algebra with bijective antipode. Let

$$(0.4) \quad W = V_1 \oplus \cdots \oplus V_\theta$$

be a direct sum of finite-dimensional irreducible Yetter-Drinfeld modules over H . Assume for simplicity that the adjoint action of $\mathcal{B}(W)$ on itself is locally finite. We fix an index i , $1 \leq i \leq \theta$. Let

$$(0.5) \quad a_{ij} = 1 - \text{top degree of } \text{ad } \mathcal{B}(V_i)(V_j), \quad i \neq j, \text{ and } a_{ii} = 2.$$

Then $(a_{ij})_{1 \leq i, j \leq \theta}$ is a generalized Cartan matrix attached to W ; note that a version of the quantum Serre relations holds by definition. We define

$V'_i = V_i^*$, V'_j as the top homogeneous component of $\text{ad } \mathcal{B}(V_i)(V_j)$ if $i \neq j$, and

$$(0.6) \quad W' = V'_1 \oplus \cdots \oplus V'_\theta.$$

Let \mathcal{K} be the algebra of coinvariant elements of $\mathcal{B}(W)$ with respect to the right coaction of $\mathcal{B}(V_i)$, and $\#$ denotes the smash product introduced in Definition 2.5. Our first main result is the key step for the construction of the family of Cartan matrices generalizing [H06, Prop. 1] where all V_i are one-dimensional.

Theorem 1. *There is an isomorphism*

$$(0.7) \quad \mathcal{B}(W') \simeq \mathcal{K} \# \mathcal{B}(V_i^*).$$

In particular, if $\dim \mathcal{B}(W) < \infty$, then $\dim \mathcal{B}(W) = \dim \mathcal{B}(W')$.

The assignment $W \mapsto W'$ is a generalized i -th reflection. Theorem 1 allows to find by iterated reflections a class of new Nichols algebras $\mathcal{B}(W')$ of the same dimension as $\mathcal{B}(W)$. This defines an equivalence relation between non-isomorphic Nichols algebras. The collection of generalized Cartan matrices we are looking for consists of the generalized Cartan matrices of the Nichols algebras in the equivalence class of $\mathcal{B}(W)$. We are now in a position to define real roots of $\mathcal{B}(W)$, see Section 3.5.

In order to prove Theorem 1, we have to overcome several difficulties. The proof of [H06, Prop. 1] depends on the existence of PBW-bases shown by Kharchenko [Kh99]. In our case where not all the V_i are one-dimensional such bases do not exist in general. Another difficulty in the general case is to prove irreducibility of the Yetter-Drinfeld modules V'_i in (0.6). Our proof of Theorem 1 can not rely on the usual characterization of a Nichols algebra as a braided Hopf algebra with special properties, because it does not seem possible to describe the comultiplication of $\mathcal{K} \# \mathcal{B}(V_i^*)$ explicitly. Instead, we use a new characterization of Nichols algebras in terms of braided derivations, see Theorem 2.9. This new characterization is a powerful tool to deal with Nichols algebras; we expect many applications of it.

6. Having defined the collection of generalized Cartan matrices and reflections attached to our W , the following questions arise:

- (A) To develop a theory of generalized root systems that correspond to our collections of generalized Cartan matrices, including classifications of suitable classes of them.
- (B) To obtain answers to questions (a) and (b) in page 3 on the Nichols algebra $\mathcal{B}(W)$ from the structure of its generalized root systems.

These matters are out of the scope of the present paper. In [HS08] the generalized root system of $\mathcal{B}(W)$ is defined (under the restriction that H is semisimple or more generally that all finite tensor powers of W are semisimple). These root systems satisfy the axioms introduced in [HY08] and studied in [CH08a, CH08b].

We present however a partial answer to question (B). Let us say that W is *standard* if the generalized Cartan matrix $(a'_{ij})_{1 \leq i, j \leq \theta}$ corresponding to W' coincides with the generalized Cartan matrix $(a_{ij})_{1 \leq i, j \leq \theta}$ corresponding to W , for all W' obtained from W by finitely many reflections.

Theorem 2. *If W is standard and $\dim \mathcal{B}(W) < \infty$, then the generalized Cartan matrix is of finite type.*

See Thm. 3.29. By [HS08, Corollary 7.4] the converse of Theorem 2 is true, that is, if W is standard, $\dim \mathcal{B}(V'_i) < \infty$ for all i and all W' obtained from W by iterated reflections, and if $(a_{ij})_{1 \leq i, j \leq \theta}$ is of finite type, then $\dim \mathcal{B}(W) < \infty$. Using the results of the present paper a necessary and sufficient criterion for $\dim \mathcal{B}(W) < \infty$ is given in the general non-standard case in [HS08, Theorem 7.3].

7. There is at the present moment no general method to deal with questions (a) and (b) for *irreducible* Yetter-Drinfeld modules over a finite non-abelian group. In fact, we know very few examples with finite dimension. The first examples, calculated in 1995, correspond to the transpositions in \mathbb{S}_n , $n = 3, 4, 5$ [MS00]. As an application of Theorem 2 for \mathbb{S}_3 and \mathbb{S}_4 , we prove that $\mathcal{B}(W)$ is infinite-dimensional if W is not irreducible. (In [HS08] this result is generalized to all finite simple groups and to all symmetric groups \mathbb{S}_n , $n \geq 3$.) This allows to conclude the classifications of finite-dimensional pointed Hopf algebras over \mathbb{S}_3 , Theorem 4.5, and of finite-dimensional Nichols algebras over \mathbb{S}_4 , Theorem 4.7. The group \mathbb{S}_3 is the first non-abelian group G where the classification of finite-dimensional pointed Hopf algebras with coradical $\mathbb{k}G$ is known, and where a Hopf algebra other than the group algebra exists. Recently, some groups that admit no finite-dimensional pointed Hopf algebra except the group algebra were found: \mathbb{A}_n , $n \geq 5$, $n \neq 6$ [AF07, AFGV08] and $SL(2, q)$ with q even [FGV]. Theorems 4.5 and 4.7 can be rephrased in terms of racks, giving rise to new techniques to establish that some Nichols algebras have infinite dimension [AF08]. These techniques have been applied in [AFZ08, AFGV08, AFGV09].

8. The paper is organized in four sections, besides this introduction. In Sect. 1 we collect several well-known results that will be used later on. In Sect. 2 we use quantum differential operators to give a new characterization of Nichols algebras. Sect. 3 is the bulk of the paper: We construct the reflection of a semisimple Yetter-Drinfeld module satisfying some hypothesis (for instance, having finite-dimensional Nichols algebra), discuss the notion of “standard” modules, and prove our main theorems. In Sect. 4 we state a few general consequences of the theory in the previous sections, and then prove the classification results for \mathbb{S}_3 and \mathbb{S}_4 alluded above. We also include a result on Nichols algebras over the dihedral group \mathbb{D}_n with n odd.

In the paper H denotes a Hopf algebra with bijective antipode \mathcal{S} .

1. PRELIMINARIES

1.1. Notation. Let \mathbb{k} be a field. All vector spaces, algebras, coalgebras, Hopf algebras, unadorned tensor products and unadorned Hom spaces are over \mathbb{k} . If V is a vector space and $n \in \mathbb{N}$, then $V^{\otimes n}$ or $T^n(V)$ denote the n -fold tensor product of V with itself. We use the notation $\langle \cdot, \cdot \rangle : \text{Hom}(V, \mathbb{k}) \times V \rightarrow \mathbb{k}$ for the standard evaluation. We identify $\text{Hom}(V, \mathbb{k}) \otimes \text{Hom}(V, \mathbb{k})$ with a subspace of $\text{Hom}(V \otimes V, \mathbb{k})$ by the recipe

$$\langle f \otimes g, v \otimes w \rangle = \langle f, w \rangle \langle g, v \rangle$$

for $f, g \in \text{Hom}(V, \mathbb{k})$, $v, w \in V$. Consequently, we identify $\text{Hom}(V, \mathbb{k})^{\otimes n}$ with a subspace of $\text{Hom}(V^{\otimes n}, \mathbb{k})$, $n \in \mathbb{N}$, via

$$(1.1) \quad \langle f_n \otimes \dots \otimes f_1, v_1 \otimes \dots \otimes v_n \rangle = \prod_{1 \leq i \leq n} \langle f_i, v_i \rangle,$$

for $f_1, \dots, f_n \in \text{Hom}(V, \mathbb{k})$, $v_1, \dots, v_n \in V$.

Let $\theta \in \mathbb{N}$ and let $\mathbb{I} = \{1, \dots, \theta\}$. Let $V = \bigoplus_{\alpha \in \mathbb{Z}^\theta} V_\alpha$ be a \mathbb{Z}^θ -graded vector space. If $\alpha = (n_1, \dots, n_\theta) \in \mathbb{Z}^\theta$, then let $\text{pr}_\alpha = \text{pr}_{n_1, \dots, n_\theta} : V \rightarrow V_\alpha$ denote the projection associated to this direct sum. We identify $\text{Hom}(V_\alpha, \mathbb{k})$ with a subspace of $\text{Hom}(V, \mathbb{k})$ via the transpose of pr_α . The graded dual of V is

$$(1.2) \quad V^{\text{gr-dual}} = \bigoplus_{\alpha \in \mathbb{Z}^\theta} \text{Hom}(V_\alpha, \mathbb{k}) \subset \text{Hom}(V, \mathbb{k}).$$

If $V = \bigoplus_{\alpha \in \mathbb{Z}^\theta} V_\alpha$ is a \mathbb{Z}^θ -graded vector space, then the support of V is $\text{supp } V := \{\alpha \in \mathbb{Z}^\theta \mid V_\alpha \neq 0\}$.

Let C be a coassociative coalgebra. Let $\Delta^n : C \rightarrow C^{\otimes(n+1)}$ denote the n -th iterated comultiplication of C . Let $G(C)$ denote the set of group-like elements of C . If $g, h \in G(C)$, then let $\mathcal{P}_{g,h}(C)$ denote the space $\{x \in C \mid \Delta(x) = g \otimes x + x \otimes h\}$ of g, h skew-primitive elements of C . If C is a braided bialgebra, then $\mathcal{P}(C) := \mathcal{P}_{1,1}(C)$. The category of left (resp. right) C -comodules is denoted ${}^C\mathcal{M}$, resp. \mathcal{M}^C . We use Sweedler's notation for the comultiplication of C : If $x \in C$, then $\Delta(x) = x_{(1)} \otimes x_{(2)}$. Similarly, the coaction of a left C -comodule M is denoted $\delta(m) = m_{(-1)} \otimes m_{(0)} \in C \otimes M$, $m \in M$.

Remark 1.1. The dual vector space $C^* = \text{Hom}(C, \mathbb{k})$ is an algebra with the convolution product: $\langle fg, c \rangle = \langle g, c_{(1)} \rangle \langle f, c_{(2)} \rangle$, cf. (1.1), for $f, g \in C^*$, $c \in C$. The reader should be warned that usually one writes $C^{*\text{op}}$ for this algebra, see [Mo93, Sect. 1.4.1]. With our convention – forced by (1.1) – any left C -comodule becomes a *left* C^* -module by

$$(1.3) \quad f \cdot m = \langle f, m_{(-1)} \rangle m_{(0)},$$

$f \in C^*$, $m \in M$. Indeed, if also $g \in C^*$, then

$$\begin{aligned} f \cdot (g \cdot m) &= \langle g, m_{(-1)} \rangle f \cdot m_{(0)} = \langle f, m_{(-1)} \rangle \langle g, m_{(-2)} \rangle m_{(0)} \\ &= \langle fg, m_{(-1)} \rangle m_{(0)} = (fg) \cdot m. \end{aligned}$$

Recall that a graded coalgebra is a coalgebra C provided with a grading $C = \bigoplus_{m \in \mathbb{N}_0} C^m$ such that $\Delta(C^m) \subset \bigoplus_{i+j=m} C^i \otimes C^j$. Then the graded dual $C^{\text{gr-dual}}$ is a subalgebra of C^* .

Let $\Delta_{i,j} : C^m \rightarrow C^i \otimes C^j$ denote the composition $\text{pr}_{i,j} \Delta$, where $m = i + j$. More generally, if $i_1, \dots, i_n \in \mathbb{N}_0$ and $i_1 + \dots + i_n = m$, then Δ_{i_1, \dots, i_n} is the composition $\text{pr}_{i_1, \dots, i_n} \Delta^{n-1}$:

$$(1.4) \quad \begin{array}{ccc} C^m & \xrightarrow{\Delta^{n-1}} & \bigoplus_{j_1 + \dots + j_n = m} C^{j_1} \otimes \dots \otimes C^{j_n} \\ \Delta_{i_1, \dots, i_n} \downarrow & \swarrow \text{pr}_{i_1, \dots, i_n} & \\ C^{i_1} \otimes \dots \otimes C^{i_n} & & \end{array}$$

Remark 1.2. Let C be a coalgebra, let $M \in {}^C\mathcal{M}$ and let $Z \subset M$ be a vector subspace. Then the subcomodule generated by Z is

$$(1.5) \quad C^* \cdot Z = \mathbb{k}\text{-span of } \{ \langle f, z_{(-1)} \rangle z_{(0)} \mid z \in Z, f \in C^* \}.$$

If $C = \bigoplus_{m \in \mathbb{N}_0} C^m$ is a graded coalgebra, then

$$(1.6) \quad C^* \cdot Z = \mathbb{k}\text{-span of } \{ \langle f, z_{(-1)} \rangle z_{(0)} \mid z \in Z, f \in C^{\text{gr-dual}} \}.$$

Proof. Clearly, (1.5) is the subcomodule generated by Z . Assume that $\dim Z < \infty$. Then there exists $m \in \mathbb{N}$ such that $\delta(Z) \subset \bigoplus_{0 \leq n \leq m} C^n \otimes M$. Therefore, in (1.5) it suffices to take

$$f \in (\bigoplus_{n > m} C^n)^\perp \simeq (\bigoplus_{0 \leq n \leq m} C^n)^* \subset \bigoplus_{n \geq 0} (C^n)^*.$$

If $\dim Z$ is arbitrary, then

$$C^* \cdot Z = C^* \cdot \left(\sum_{Z' \subset Z : \dim Z' < \infty} Z' \right) = \sum_{Z' \subset Z \mid \dim Z' < \infty} (C^* \cdot Z'),$$

proving the assertion. \square

1.2. Yetter-Drinfeld modules. Our reference for the theory of Hopf algebras is [Mo93]. Recall that H is a Hopf algebra with bijective antipode \mathcal{S} . The adjoint representation of H on itself is the algebra map $\text{ad} : H \rightarrow \text{End } H$, $\text{ad } x(y) = x_{(1)}y\mathcal{S}(x_{(2)})$, $x, y \in H$. Then

$$(1.7) \quad \text{ad } x(yy') = \text{ad}(x_{(1)})(y) \text{ad}(x_{(2)})(y'),$$

$x, y, y' \in H$. That is, H is a left H -module algebra via the adjoint.

Let ${}^H_H\mathcal{YD}$ be the category of Yetter-Drinfeld modules over H ; $V \in {}^H_H\mathcal{YD}$ is a left H -module and a left H -comodule such that

$$(1.8) \quad \delta(h \cdot x) = h_{(1)}x_{(-1)}\mathcal{S}(h_{(3)}) \otimes h_{(2)} \cdot x_{(0)},$$

$h \in H$, $x \in V$. It is well-known that ${}^H_H\mathcal{YD}$ is a braided tensor category, with braiding $c_{V,W} : V \otimes W \rightarrow W \otimes V$, $c_{V,W}(v \otimes w) = v_{(-1)} \cdot w \otimes v_{(0)}$, $V, W \in {}^H_H\mathcal{YD}$, $v \in V$, $w \in W$. We record that the inverse braiding is given by

$$(1.9) \quad c_{V,W}^{-1}(v \otimes w) = w_{(0)} \otimes \mathcal{S}^{-1}(w_{(-1)}) \cdot v,$$

$V, W \in {}^H_H\mathcal{YD}$, $v \in V$, $w \in W$.

Remark 1.3. Let $V \in {}^H_H\mathcal{YD}$.

(i) If $U \subset V$ is an H -submodule, then the subcomodule $H^* \cdot U$ generated by U is a Yetter-Drinfeld submodule of V .

(ii) If $T \subset V$ is an H -subcomodule, then the submodule $H \cdot T$ generated by T is a Yetter-Drinfeld submodule of V .

Proof. (i). If $u \in U$, $f \in H^*$ and $h \in H$, then

$$h \cdot (\langle f, u_{(-1)} \rangle u_{(0)}) = \langle f, \mathcal{S}(h_{(1)})(h_{(2)} \cdot u)_{(-1)}h_{(3)} \rangle (h_{(2)} \cdot u)_{(0)} \in H^* \cdot U$$

by (1.8). (ii) is also a direct consequence of (1.8). \square

Let $V \in {}^H_H\mathcal{YD}$ be finite-dimensional. The left and right duals of V are respectively denoted *V and V^* . As vector spaces, ${}^*V = V^* = \text{Hom}(V, \mathbb{k})$. Their structures of Yetter-Drinfeld modules are determined by requiring that the following natural maps are morphisms in ${}^H_H\mathcal{YD}$:

$$\begin{aligned} \text{ev} : V^* \otimes V &\rightarrow \mathbb{k}, & \text{coev} : \mathbb{k} &\rightarrow V \otimes V^*, \\ \text{ev} : V \otimes {}^*V &\rightarrow \mathbb{k}, & \text{coev} : \mathbb{k} &\rightarrow {}^*V \otimes V, \end{aligned}$$

cf. [BK00, Def. 2.1.1]. Thus V^* has action and coaction given by

$$(1.10) \quad \langle h \cdot f, v \rangle = \langle f, \mathcal{S}(h) \cdot v \rangle,$$

$$(1.11) \quad f_{(-1)} \langle f_{(0)}, v \rangle = \mathcal{S}^{-1}(v_{(-1)}) \langle f, v_{(0)} \rangle,$$

$f \in V^*$, $v \in V$. Albeit evident, we record that (1.11) is equivalent to

$$(1.12) \quad \mathcal{S}(f_{(-1)}) \langle f_{(0)}, v \rangle = v_{(-1)} \langle f, v_{(0)} \rangle,$$

$f \in V^*$, $v \in V$. Notice that (1.10) provides $V^* = \text{Hom}(V, \mathbb{k})$ with an H -module structure, regardless of whether $\dim V$ is finite or not.

It is easy to see that $T^n(V^*)$ is a Yetter-Drinfeld submodule of $(T^n(V))^*$ via the identification (1.1). Also, the evaluation $\langle \cdot, \cdot \rangle : V^* \times V \rightarrow \mathbb{k}$ satisfies

$$(1.13) \quad \langle c_{V^*}(f \otimes g), v \otimes w \rangle = \langle f \otimes g, c_V(v \otimes w) \rangle,$$

$f, g \in V^*$, $v, w \in V$.

Proof. We compute

$$\begin{aligned}\langle c_{V^*}(f \otimes g), v \otimes w \rangle &= \langle f_{(-1)} \cdot g \otimes f_{(0)}, v \otimes w \rangle = \langle f_{(0)}, v \rangle \langle f_{(-1)} \cdot g, w \rangle \\ &= \langle f_{(0)}, v \rangle \langle g, \mathcal{S}(f_{(-1)}) \cdot w \rangle = \langle f, v_{(0)} \rangle \langle g, v_{(-1)} \cdot w \rangle \\ &= \langle f \otimes g, v_{(-1)} \cdot w \otimes v_{(0)} \rangle = \langle f \otimes g, c_V(v \otimes w) \rangle.\end{aligned}$$

□

Analogously, *V has action and coaction given by $\langle v, h \cdot f \rangle = \langle \mathcal{S}^{-1}(h) \cdot v, f \rangle$, $f_{(-1)} \langle v, f_{(0)} \rangle = \mathcal{S}(v_{(-1)}) \langle v_{(0)}, f \rangle$, $f \in {}^*V$, $v \in V$.

Remark 1.4. One has $V \simeq V^{**}$ for any finite-dimensional $V \in {}^H_H\mathcal{YD}$ [BK00, (2.2.6)]. Explicitly, if we identify V and V^{**} as vector spaces via the map $v \mapsto \varphi_v$, where $\langle \varphi_v, f \rangle := \langle f, v \rangle$ for all $f \in V^*$ and $v \in V$, then the isomorphism $\psi_V : V^{**} \rightarrow V$ in ${}^H_H\mathcal{YD}$ and its inverse $\phi_V := \psi_V^{-1}$ are given by

$$(1.14) \quad \psi_V(\varphi_v) = \mathcal{S}^{-2}(v_{(-1)}) \cdot v_{(0)},$$

$$(1.15) \quad \phi_V(v) = \mathcal{S}((\varphi_v)_{(-1)}) \cdot (\varphi_v)_{(0)}, \quad v \in V.$$

Further, (1.10) and (1.11) imply that

$$(1.16) \quad \begin{aligned}\delta(\varphi_v) &= \mathcal{S}^{-2}(v_{(-1)}) \otimes \varphi_{v_{(0)}}, \\ \langle \phi_V(v), f \rangle &= \langle v_{(-1)} \cdot f, v_{(0)} \rangle.\end{aligned}$$

1.3. Smash coproduct. We shall need later the following well-known facts. Let $C \in {}^H\mathcal{M}$ be a left comodule coalgebra— that is, the comultiplication of C is a comodule map. Let us denote the comultiplication of C by the following variation of Sweedler's notation: If $c \in C$, then $\Delta(c) = c^{(1)} \otimes c^{(2)}$. Let $C \# H$ be the corresponding smash coproduct: This is the vector space $C \otimes H$ (with generic element $c \# h$) with comultiplication

$$(1.17) \quad \Delta(c \# h) = c^{(1)} \# (c^{(2)})_{(-1)} h_{(1)} \otimes (c^{(2)})_{(0)} \# h_{(2)},$$

$c \in C$, $h \in H$. Let $p_C = \text{id} \otimes \varepsilon : C \# H \rightarrow C$ and $p_H = \varepsilon \otimes \text{id} : C \# H \rightarrow H$ be the canonical coalgebra projections. Let $\tau : H \otimes C \rightarrow C \otimes H$ be given by

$$\tau(h \otimes c) = c_{(0)} \otimes \mathcal{S}^{-1}(c_{(-1)})h, \quad h \in H, c \in C.$$

Lemma 1.5. *Let $M \in {}^C \# H \mathcal{M}$ with coaction $\delta_{C \# H}$. Hence also $M \in {}^C \mathcal{M}$ with coaction $\delta_C = (p_C \otimes \text{id})\delta_{C \# H}$ and $M \in {}^H \mathcal{M}$ with coaction $\delta_H = (p_H \otimes \text{id})\delta_{C \# H}$. Then the following hold.*

- (i) $\delta_{C \# H} = (\text{id} \otimes \delta_H)\delta_C = (\tau \otimes \text{id})(\text{id} \otimes \delta_C)\delta_H$.
- (ii) *If $N \subset M$ is both a C -subcomodule and an H -subcomodule, then it is a $C \# H$ -subcomodule.*

(iii) If $Z \subset M$ is an H -subcomodule, then the C -subcomodule generated by Z is a $C \# H$ -subcomodule.

Proof. Let $m \in M$ and write $\delta_{C \# H}(m) = m_{(C, -1)} \# m_{(H, -1)} \otimes m_{(0)}$. We spell out the coassociativity in this notation:

$$\begin{aligned}
 (1.18) \quad & m_{(C, -1)} \# m_{(H, -1)} \otimes m_{(0, C, -1)} \# m_{(0, H, -1)} \otimes m_{(0, 0)} \\
 &= (m_{(C, -1)})^{(1)} \# ((m_{(C, -1)})^{(2)})_{(-1)} (m_{(H, -1)})_{(1)} \\
 &\quad \otimes ((m_{(C, -1)})^{(2)})_{(0)} \# (m_{(H, -1)})_{(2)} \otimes m_{(0)}.
 \end{aligned}$$

Applying $p_C \otimes p_H \otimes \text{id}$ to (1.18), we get

$$\begin{aligned}
 (\text{id} \otimes \delta_H) \delta_C(m) &= m_{(C, -1)} \varepsilon(m_{(H, -1)}) \# \varepsilon(m_{(0, C, -1)}) m_{(0, H, -1)} \otimes m_{(0, 0)} \\
 &= m_{(C, -1)} \# m_{(H, -1)} \otimes m_{(0)} = \delta_{C \# H}(m).
 \end{aligned}$$

Applying $(\tau \otimes \text{id})(p_H \otimes p_C \otimes \text{id})$ to (1.18), we get

$$\begin{aligned}
 (\tau \otimes \text{id})(\text{id} \otimes \delta_C) \delta_H(m) &= \tau((m_{(C, -1)})_{(-1)} m_{(H, -1)} \otimes (m_{(C, -1)})_{(0)}) \otimes m_{(0)} \\
 &= (m_{(C, -1)})_{(0)} \otimes \mathcal{S}^{-1}((m_{(C, -1)})_{(-1)}) (m_{(C, -1)})_{(-2)} m_{(H, -1)} \otimes m_{(0)} \\
 &= \delta_{C \# H}(m).
 \end{aligned}$$

Now (ii) follows from the first equality in Lemma 1.5 (i). Finally, the equality of the first and third expressions in Lemma (1.5) (i) gives that the $C \# H$ -subcomodule generated by Z is contained in (and hence it coincides with) the C -subcomodule generated by Z . This gives (iii). \square

1.4. Braided Hopf algebras and bosonization. We briefly summarize results from [Ra85], see also [Ma94]. Let A be a Hopf algebra provided with Hopf algebra maps $\pi : A \rightarrow H$, $\iota : H \rightarrow A$, such that $\pi \iota = \text{id}_H$. In other words, we have a commutative diagram in the category of Hopf algebras:

$$\begin{array}{ccc}
 & & H \\
 & \nearrow \iota & \parallel \\
 A & \xrightarrow{\pi} & H.
 \end{array}$$

Let $R = A^{\text{co}H} = \{a \in A \mid (\text{id} \otimes \pi_H) \Delta(a) = a \otimes 1\}$. Then R is a braided Hopf algebra in ${}^H_H \mathcal{YD}$. Following the notation in Subsection 1.3, let $\Delta(r) = r^{(1)} \otimes r^{(2)}$ denote the coproduct of $r \in R$ (or any other braided Hopf algebra). Explicitly, R is a subalgebra of A , and

$$\begin{aligned}
 (1.19) \quad & h \cdot r = h_{(1)} r \mathcal{S}(h_{(2)}), \\
 & r_{(-1)} \otimes r_{(0)} = \pi(r_{(1)}) \otimes r_{(2)}, \\
 & r^{(1)} \otimes r^{(2)} = \vartheta_R(r_{(1)}) \otimes r_{(2)},
 \end{aligned}$$

$r \in R$, $h \in H$. Here $\vartheta_R : A \rightarrow R$ is the map defined by

$$(1.20) \quad \vartheta_R(a) = a_{(1)}\iota\pi(\mathcal{S}(a_{(2)})),$$

$a \in A$. It can be easily shown that

$$(1.21) \quad \vartheta_R(rh) = r\varepsilon(h), \quad \vartheta_R(hr) = h \cdot r$$

for $r \in R$, $h \in H$. Reciprocally, let R be a braided Hopf algebra in ${}^H_H\mathcal{YD}$. A construction discovered by Radford, and interpreted in terms of braided categories by Majid, produces a Hopf algebra $R\#H$ from R . We call $R\#H$ the bosonization of R . As a vector space, $R\#H = R \otimes H$; if $r\#h := r \otimes h$, $r \in R$, $h \in H$, then the multiplication and comultiplication of $R\#H$ are given by

$$(1.22) \quad \begin{aligned} (r\#h)(s\#f) &= r(h_{(1)} \cdot s)\#h_{(2)}f, \\ \Delta(r\#h) &= r^{(1)}\#(r^{(2)})_{(-1)}h_{(1)} \otimes (r^{(2)})_{(0)}\#h_{(2)}. \end{aligned}$$

The maps

$$\pi_H : R\#H \rightarrow H \text{ and } \iota : H \rightarrow R\#H, \quad \pi_H(r\#h) = \varepsilon(r)h, \quad \iota(h) = 1\#h,$$

$r \in R$, $h \in H$, are Hopf algebra homomorphisms; we identify H with the image of ι . Hence

$$(1.23) \quad r_{(1)} \otimes r_{(2)} = r^{(1)}(r^{(2)})_{(-1)} \otimes (r^{(2)})_{(0)},$$

$r \in R$. The map $p_R : R\#H \rightarrow R$, $p_R(r\#h) = r\varepsilon(h)$, $r \in R$, $h \in H$, is a coalgebra homomorphism – see page 10. We shall write rh instead of $r\#h$, $r \in R$, $h \in H$. The antipodes \mathcal{S}_R of R and $\mathcal{S} = \mathcal{S}_{R\#H}$ of $R\#H$ are related by

$$(1.24) \quad \begin{aligned} \mathcal{S}_R(r) &= r_{(-1)}\mathcal{S}(r_{(0)}), \\ \mathcal{S}(r) &= \mathcal{S}(r_{(-1)})\mathcal{S}_R(r_{(0)}), \end{aligned}$$

$r \in R$. The antipode \mathcal{S}_R is a morphism of Yetter-Drinfeld modules. Let μ be the multiplication of R and $c \in \text{End}(R \otimes R)$ be the braiding. Then \mathcal{S}_R is anti-multiplicative and anti-comultiplicative in the following sense:

$$(1.25) \quad \begin{aligned} \mathcal{S}_R\mu &= \mu(\mathcal{S}_R \otimes \mathcal{S}_R)c = \mu c(\mathcal{S}_R \otimes \mathcal{S}_R), \\ \Delta\mathcal{S}_R &= (\mathcal{S}_R \otimes \mathcal{S}_R)c\Delta = c(\mathcal{S}_R \otimes \mathcal{S}_R)\Delta, \end{aligned}$$

see for instance [AG99, 1.2.2]. The adjoint representation of R on itself is the algebra map $\text{ad}_c : R \rightarrow \text{End } R$, $\text{ad}_c x(y) = \mu(\mu \otimes \mathcal{S})(\text{id} \otimes c)(\Delta \otimes \text{id})(x \otimes y)$, $x, y \in R$. That is,

$$(1.26) \quad \text{ad}_c x(y) = x^{(1)}[(x^{(2)})_{(-1)} \cdot y]\mathcal{S}((x^{(2)})_{(0)}) = \text{ad } x(y)$$

for all $x, y \in R$, where the second equality follows immediately from (1.19) and (1.24). If $x \in \mathcal{P}(R)$, then

$$(1.27) \quad \text{ad}_c x(y) = xy - (x_{(-1)} \cdot y)x_{(0)}$$

for all $y \in R$. Similarly, define

$$\text{ad}_{c^{-1}} x(y) = xy - y_{(0)}(\mathcal{S}^{-1}(y_{(-1)}) \cdot x)$$

for $x \in \mathcal{P}(R)$, $y \in R$. We record the next well-known remark for further reference.

Remark 1.6. The space of primitive elements $\mathcal{P}(R)$ is a Yetter-Drinfeld submodule of R . \square

The next consequences of (1.25) will be used later.

Lemma 1.7. (i) *Let $x \in \mathcal{P}(R)$, $y \in R$. Then*

$$(1.28) \quad \text{ad}_c x(\mathcal{S}_R(y)) = \mathcal{S}_R(\text{ad}_{c^{-1}} x(y)).$$

(ii) *Let X be a Yetter-Drinfeld submodule of R and let K be the subalgebra generated by X . Then $\mathcal{S}_R(K)$ is the subalgebra generated by $\mathcal{S}_R(X)$.*

Proof. Since $\mathcal{S}_R(x) = -x$, (i) follows directly from (1.25): $\text{ad}_c x(\mathcal{S}_R(y)) = -\mu(\mathcal{S}_R \otimes \mathcal{S}_R)(\text{id} - c)(x \otimes y) \stackrel{(1.25)}{=} -\mathcal{S}_R \mu(c^{-1} - \text{id})(x \otimes y) = \mathcal{S}_R(\text{ad}_{c^{-1}} x(y))$.

(ii). If X, Y are Yetter-Drinfeld submodules of R , then XY is also a Yetter-Drinfeld submodule and $\mathcal{S}_R(XY) = \mathcal{S}_R(Y)\mathcal{S}_R(X)$ by (1.25). This implies immediately (ii). \square

Remark 1.8. Let K be a left A -module algebra, that is, K is a left H -module algebra and a left R -module such that the action \cdot of R on K satisfies equation $r \cdot (k\tilde{k}) = (r^{(1)} \cdot ((r^{(2)})_{(-1)} \cdot k))((r^{(2)})_{(0)} \cdot \tilde{k})$ for all $r \in R$, $k, \tilde{k} \in K$.

(i) The smash product $K \# A$ is a right H -comodule algebra via the coaction $(\text{id} \# \text{id} \otimes \pi)(\text{id} \# \Delta)$, with subalgebra of coinvariants $K \# R$. According to (1.17), the product in the last is given by

$$(k \# r)(k' \# r') = k(r^{(1)}(r^{(2)})_{(-1)}) \cdot k' \# (r^{(2)})_{(0)} r',$$

$k, k' \in K$, $r, r' \in R$.

(ii) The multiplication induces a linear isomorphism $R \otimes K \rightarrow K \# R$. The inverse map is given by $k \# r \mapsto r_{(2)} \otimes \mathcal{S}^{-1}(r_{(1)}) \cdot k$.

Remark 1.9. Let B be a braided bialgebra. Let B^{cop} denote the algebra B together with the comultiplication $c^{-1}\Delta$; this is a braided Hopf algebra but with the inverse braiding, see [AG99, Prop. 2.2.4]. Clearly, $\mathcal{P}(B) = \mathcal{P}(B^{\text{cop}})$.

1.5. Nichols algebras. Let $V \in {}^H_H \mathcal{YD}$. The tensor algebra $T(V)$ is a braided Hopf algebra in ${}^H_H \mathcal{YD}$. A very important example of braided Hopf algebra in ${}^H_H \mathcal{YD}$ is the Nichols algebra $\mathcal{B}(V)$ of V ; this is the quotient of $T(V)$ by a homogeneous ideal $\mathfrak{J} = \mathfrak{J}(V)$, generated by (some) homogeneous elements of degree ≥ 2 . See [AS02] for the precise definition and main properties of Nichols algebras, and the relation with pointed Hopf algebras.

Another description of the ideal $\mathfrak{J}(V)$ is as the kernel of the quantum symmetrizer introduced by Woronowicz [W89], see [S96]. Let \mathbb{B}_n be the braid group in n letters and let $\pi : \mathbb{B}_n \rightarrow \mathbb{S}_n$ be a natural projection; it admits a set-theoretical section $s : \mathbb{S}_n \rightarrow \mathbb{B}_n$ called the Matsumoto section. Let $\mathfrak{S}_n := \sum_{\sigma \in \mathbb{S}_n} s(\sigma)$. The braid group \mathbb{B}_n acts on $T^n(V)$ via c and the homogeneous component $\mathfrak{J}^n(V)$ of $\mathfrak{J}(V)$ equals $\ker \mathfrak{S}_n$. Thus $\mathcal{B}(V)$ depends (as algebra and coalgebra) only on the braiding c . We write $\mathcal{B}(V) = \mathcal{B}(V, c)$, $\mathfrak{J}(V) = \mathfrak{J}(V, c)$.

The Nichols algebra has a unique grading $\mathcal{B}(V) = \bigoplus_{n \in \mathbb{N}_0} \mathcal{B}^n(V)$ such that $\mathcal{B}^1(V) = V$, the multiplication and the comultiplication are graded, and the action and the coaction of H are homogeneous.

If $\dim V < \infty$, then there exists a bilinear form $\langle \cdot, \cdot \rangle : T(V^*) \times T(V) \rightarrow \mathbb{k}$ such that

$$(1.29) \quad \langle T^n(V^*), T^m(V) \rangle = 0, \quad n \neq m,$$

$$(1.30) \quad \langle f_n \dots f_1, x \rangle = \langle f_n \otimes \dots \otimes f_1, \Delta_{1, \dots, 1}(x) \rangle$$

for $f_1, \dots, f_n \in V^*$, $x \in T^n(V)$, $n \in \mathbb{N}_0$. It satisfies the following properties:

$$(1.31) \quad \langle fg, x \rangle = \langle f, x^{(2)} \rangle \langle g, x^{(1)} \rangle,$$

$$(1.32) \quad \langle f, xy \rangle = \langle f^{(2)}, x \rangle \langle f^{(1)}, y \rangle,$$

$$(1.33) \quad \langle h \cdot f, x \rangle = \langle f, \mathcal{S}(h) \cdot x \rangle,$$

$$(1.34) \quad f_{(-1)} \langle f_{(0)}, x \rangle = \mathcal{S}^{-1}(x_{(-1)}) \langle f, x_{(0)} \rangle$$

for all $f, g \in T(V^*)$, $x, y \in T(V)$, $h \in H$. This was first observed in [Ma93], see also [Ma95, 10.4.13]. A combination of the explicit formulas in [Ma95, 10.4.13] and [W89, Eqs. (3.25), (3.26)] shows that

$$\Delta_{1, \dots, 1} = \mathfrak{S}_n$$

for all $n \in \mathbb{N}$, that is, $\mathfrak{J}(V, c)$ is the radical of the form in the second argument. More precisely, the following holds.

Proposition 1.10. [AG99, Thm. 3.2.29] *Assume that $V \in {}^H_H \mathcal{YD}$ such that $\dim V < \infty$. Then there exists a non-degenerate bilinear form*

$$\langle \cdot, \cdot \rangle : \mathcal{B}(V^*) \times \mathcal{B}(V) \rightarrow \mathbb{k}$$

such that

$$(1.35) \quad \langle \mathcal{B}^n(V^*), \mathcal{B}^m(V) \rangle = 0, \quad n \neq m,$$

$$(1.36) \quad \langle f_n \dots f_1, x \rangle = \langle f_n \otimes \dots \otimes f_1, \Delta_{1, \dots, 1}(x) \rangle,$$

for $f_1, \dots, f_n \in V^$, $x \in \mathcal{B}^n(V)$, $n \in \mathbb{N}_0$. It satisfies (1.31), (1.32), (1.33), and (1.34) for all $f, g \in \mathcal{B}(V^*)$, $x, y \in \mathcal{B}(V)$, $h \in H$.*

This proposition tells that

$$(1.37) \quad \mathcal{B}(V)^{\text{gr-dual}} \simeq \mathcal{B}(V^*),$$

where $\mathcal{B}(V)^{\text{gr-dual}}$ is the graded dual of $\mathcal{B}(V)$, see (1.2).

Lemma 1.11. $\mathfrak{J}(V, c) = \mathfrak{J}(V, c^{-1})$ and $\mathcal{B}(V, c) \simeq \mathcal{B}(V, c^{-1})$ as algebras.

Proof. Let $\mathcal{B}(V)^{\text{cop}}$ be the opposite coalgebra, see Remark 1.9. Clearly, the algebra $\mathcal{B}(V)^{\text{cop}}$ is generated in degree one, and $\mathcal{P}(\mathcal{B}(V)^{\text{cop}}) = \mathcal{P}(\mathcal{B}(V)) = V$. Hence $\mathcal{B}(V)^{\text{cop}} \simeq \mathcal{B}(V, c^{-1})$, and $\mathfrak{J}(V, c) = \mathfrak{J}(V, c^{-1})$. \square

Lemma 1.12. Let $x = \sum_{n \geq 1} x(n) \in \mathcal{B}(V)$, with $x(n) \in \mathcal{B}^n(V)$. Assume that $x^{(1)} \otimes \text{pr}_1(x^{(2)}) = 0$. Then $x = 0$.

Proof. From $0 = x^{(1)} \otimes \text{pr}_1(x^{(2)}) = \sum_{n \geq 1} x(n)^{(1)} \otimes \text{pr}_1(x(n)^{(2)})$ we conclude that $\Delta_{n-1,1}(x(n)) = x(n)^{(1)} \otimes \text{pr}_1(x(n)^{(2)}) = 0$, since $x(n)^{(1)} \otimes \text{pr}_1(x(n)^{(2)}) \in \mathcal{B}^{n-1}(V) \otimes \mathcal{B}^1(V)$. But $\Delta_{n-1,1}$ is injective in a Nichols algebra, hence $x(n) = 0$ for all n and *a fortiori* $x = 0$. \square

For simplicity, we write $\mathcal{A}(V) = \mathcal{B}(V) \# H$ for the bosonization of $\mathcal{B}(V)$. Then $\mathcal{A}(V) = \oplus_{n \in \mathbb{N}_0} \mathcal{A}^n(V)$, where $\mathcal{A}^n(V) = \mathcal{B}^n(V) \# H$, is a graded Hopf algebra.

2. THE ALGEBRA OF QUANTUM DIFFERENTIAL OPERATORS

We now discuss two algebras of quantum differential operators that appeared frequently in the literature. For quantum groups, it seems that they were first defined in [Ka91], see also [L93, Chapter 15]. For Yetter-Drinfeld modules over finite group algebras, see [Gñ00].

2.1. The algebra of quantum differential operators. Let B be a braided bialgebra in ${}^H_H\mathcal{YD}$. Then the space of linear endomorphisms $\text{End } B$ is an associative algebra with respect to the convolution product: $T * S(b) = T(b^{(2)})S(b^{(1)})$, $T, S \in \text{End } B$, $b \in B$, a convention coherent with (1.1). Since B is a left and right comodule over itself via the comultiplication, it becomes a left and right module over B^* . If $\xi \in B^*$, then we define the quantum differential operators $\partial^L, \partial^R : B^* \rightarrow \text{End } B$ as the representations associated to those actions. That is,

$$(2.1) \quad \partial_\xi^L(b) = \langle \xi, b^{(1)} \rangle b^{(2)}, \quad \partial_\xi^R(b) = \langle \xi, b^{(2)} \rangle b^{(1)}, \quad b \in B, \xi \in B^*.$$

Let also $L, R : B \rightarrow \text{End } B$ be the left and right regular representations.

If $\xi, \zeta \in B^*$, then clearly $\partial_\xi^L \partial_\zeta^R = \partial_\xi^R \partial_\zeta^L$. Other basic properties of the quantum differential operators are stated in the next lemma.

Recall that A° denotes the Sweedler dual of an algebra A . Explicitly,

$$A^\circ = \{f \in \text{Hom}(A, \mathbb{k}) \mid \ker f \text{ contains a left ideal } I \text{ of finite codimension}\}.$$

Lemma 2.1. (i) *The maps $\partial^L : B^* \rightarrow \text{End } B$ and $\partial^R : B^{*\text{op}} \rightarrow \text{End } B$ are injective algebra homomorphisms.*

(ii) *If B is a braided Hopf algebra with bijective antipode, then the maps $\Psi^L, \Psi^R : B \otimes B^* \rightarrow \text{End } B$, $\Psi^L(b \otimes \xi) = L_b \circ \partial_\xi^L$, $\Psi^R(b \otimes \xi) = R_b \circ \partial_\xi^R$, are injective.*

(iii) *If $\xi \in B^0$ and $b, c \in B$, then*

$$(2.2) \quad \partial_\xi^L(bc) = \langle \xi^{(2)}, b^{(1)} \rangle (b^{(2)})_{(0)} \partial_{S^{-1}((b^{(2)})_{(-1)}) \cdot \xi^{(1)}}^L(c),$$

$$(2.3) \quad \partial_\xi^R(bc) = \partial_{(\xi^{(2)})_{(0)}}^R(b) \mathcal{S} \left((\xi^{(2)})_{(-1)} \right) \cdot \partial_{\xi^{(1)}}^R(c).$$

(iv) *If $\xi \in \mathcal{P}(B^0)$ and $b, c \in B$, then*

$$(2.4) \quad \partial_\xi^L(bc) = b_{(0)} \partial_{S^{-1}(b_{(-1)}) \cdot \xi}^L(c) + \partial_\xi^L(b)c,$$

$$(2.5) \quad \partial_\xi^R(bc) = b \partial_\xi^R(c) + \partial_{\xi_{(0)}}^R(b) \mathcal{S}(\xi_{(-1)}) \cdot c.$$

(v) *Let U be a Yetter-Drinfeld submodule of $\mathcal{P}(B^0)$. Let S be the subalgebra of B^0 generated by U . Then $\mathcal{D}^L(B, U) := L(B) \circ \partial^L(S)$ and $\mathcal{D}^R(B, U) := R(B) \circ \partial^R(S)$ are subalgebras of $\text{End } B$.*

Proof. (i). If $\partial_\xi^L(b) = 0$, then $\langle \xi, b \rangle = \varepsilon \partial_\xi^L(b) = 0$; thus ∂^L is injective – and similarly for ∂^R . Now, if $b \in B$, $\xi, \zeta \in B^*$, then

$$\begin{aligned} \partial_\zeta^L \partial_\xi^L(b) &= \langle \xi, b^{(1)} \rangle \partial_\zeta^L(b^{(2)}) = \langle \xi, b^{(1)} \rangle \langle \zeta, b^{(2)} \rangle b^{(3)} = \langle \zeta * \xi, b^{(1)} \rangle b^{(2)} = \partial_{\zeta * \xi}^L(b), \\ \partial_\zeta^R \partial_\xi^R(b) &= \langle \xi, b^{(2)} \rangle \partial_\zeta^R(b^{(1)}) = \langle \xi, b^{(3)} \rangle \langle \zeta, b^{(2)} \rangle b^{(1)} = \langle \xi * \zeta, b^{(2)} \rangle b^{(1)} = \partial_{\xi * \zeta}^R(b). \end{aligned}$$

(ii). Let $\sum_i b_i \otimes \xi_i \in \ker \Psi^L$, and assume that the b_i 's are linearly independent. Thus $\sum_i b_i \langle \xi_i, b^{(1)} \rangle b^{(2)} = 0$ for any $b \in B$. Therefore

$$\sum_i b_i \langle \xi_i, b \rangle = \sum_i b_i \langle \xi_i, b^{(1)} \rangle b^{(2)} \mathcal{S}_B(b^{(3)}) = 0 \implies \langle \xi_i, b \rangle = 0$$

for all i and $b \in B$; hence $\xi_i = 0$ for all i . The argument for Ψ^R is similar.

(iii). We compute

$$\begin{aligned} \partial_\xi^L(bc) &= \langle \xi, (bc)^{(1)} \rangle (bc)^{(2)} = \langle \xi^{(1)} \otimes \xi^{(2)}, b^{(1)} \otimes (b^{(2)})_{(-1)} \cdot c^{(1)} \rangle (b^{(2)})_{(0)} c^{(2)} \\ &= \langle \xi^{(2)}, b^{(1)} \rangle (b^{(2)})_{(0)} \langle \xi^{(1)}, (b^{(2)})_{(-1)} \cdot c^{(1)} \rangle c^{(2)} \\ &\stackrel{(1.10)}{=} \langle \xi^{(2)}, b^{(1)} \rangle (b^{(2)})_{(0)} \langle S^{-1}((b^{(2)})_{(-1)}) \cdot \xi^{(1)}, c^{(1)} \rangle c^{(2)}; \\ \partial_\xi^R(bc) &= \langle \xi, (bc)^{(2)} \rangle (bc)^{(1)} = \langle \xi^{(1)} \otimes \xi^{(2)}, (b^{(2)})_{(0)} \otimes c^{(2)} \rangle b^{(1)} (b^{(2)})_{(-1)} \cdot c^{(1)} \\ &= \langle \xi^{(2)}, (b^{(2)})_{(0)} \rangle b^{(1)} \langle \xi^{(1)}, c^{(2)} \rangle (b^{(2)})_{(-1)} \cdot c^{(1)} \\ &\stackrel{(1.12)}{=} \langle (\xi^{(2)})_{(0)}, b^{(2)} \rangle b^{(1)} \langle \xi^{(1)}, c^{(2)} \rangle \mathcal{S} \left((\xi^{(2)})_{(-1)} \right) \cdot c^{(1)}. \end{aligned}$$

Now (iv) follows at once from (iii). Next, (2.4) and (2.5) say that

$$(2.6) \quad \partial_\xi^L \circ L_b = L_{b_{(0)}} \circ \partial_{s^{-1}(b_{(-1)}) \cdot \xi}^L + L_{\partial_\xi^L(b)},$$

$$(2.7) \quad \partial_\xi^R \circ R_c = R_{\partial_\xi^R(c)} + R_{s(\xi_{(-1)}) \cdot c} \circ \partial_{\xi_{(0)}}^R$$

for $\xi \in \mathcal{P}(B^o)$, $b, c \in B$. These equalities imply (v). \square

Examples 2.2. (i). If B is a usual bialgebra, then the generalized Leibniz rules (2.2) and (2.3) simply say that

$$\partial_\xi^L(bc) = \partial_{\xi^{(2)}}^L(b) \partial_{\xi^{(1)}}^L(c), \quad \partial_\xi^R(bc) = \partial_{\xi^{(2)}}^R(b) \partial_{\xi^{(1)}}^R(c).$$

(ii). Let $W \in {}^H_H\mathcal{YD}$ be finite-dimensional and let $B = \mathcal{B}(W)$. By Prop. 1.10, there exists an embedding $\mathcal{B}(W^*) \rightarrow \mathcal{B}(W)^*$ and we can consider the algebras of *quantum differential operators*

$$\begin{aligned} \mathcal{D}^L(W) &:= \mathcal{D}^L(\mathcal{B}(W), W^*) = L(\mathcal{B}(W)) \circ \partial^L(\mathcal{B}(W^*)), \\ \mathcal{D}^R(W) &:= \mathcal{D}^R(\mathcal{B}(W), W^*) = R(\mathcal{B}(W)) \circ \partial^R(\mathcal{B}(W^*)); \end{aligned}$$

these are subalgebras of $\text{End } B$ by Lemma 2.1(v).

Let $x \in \mathcal{B}(W)$ be homogeneous of degree p . Let us write in this case

$$\Delta(x) = x \otimes 1 + 1 \otimes x + \sum_{0 < r < p} x'_r \otimes x''_{p-r}.$$

Here we use a symbolic notation with $x'_r \in \mathcal{B}^r(W)$, $x''_{p-r} \in \mathcal{B}^{p-r}(W)$. If $f \in W^*$ and $p > 1$, then $\partial_f^L(x) = \langle f, x'_1 \rangle x''_{p-1}$, $\partial_f^R(x) = x'_{p-1} \langle f, x''_1 \rangle$. Also, $\partial_f^L(w) = \langle f, w \rangle = \partial_f^R(w)$ for $w \in W$.

The following fact is well-known and goes back essentially to [N78]: If $x \in \mathcal{B}(W)$ and $\partial_f^L(x) = 0$ for all $f \in W^*$, then $x \in \mathbb{k}$.

(iii). Let W and $\mathcal{B}(W)$ as in (ii). Assume that W admits a basis v_1, \dots, v_θ such that $\delta(v_i) = g_i \otimes v_i$, for some $g_i \in G(H)$, $1 \leq i \leq \theta$. Let f_1, \dots, f_θ be the dual basis; then $\delta(f_i) = g_i^{-1} \otimes f_i$, $1 \leq i \leq \theta$. Set $\partial_i = \partial_{f_i}^R$. Then

$$\partial_i(bc) = b \partial_i(c) + \partial_i(b) g_i \cdot c, \quad b, c \in \mathcal{B}(W).$$

Similarly, let $\text{Alg}(H, \mathbb{k})$ be the group of algebra homomorphisms from H to \mathbb{k} ; it acts on B by $\chi \cdot b = \langle \chi, b_{(-1)} \rangle b_{(0)}$. Suppose that W admits a basis v_1, \dots, v_θ such that $h \cdot v_i = \chi_i(h) v_i$, for some $\chi_i \in \text{Alg}(H, \mathbb{k})$, $1 \leq i \leq \theta$. Let f_1, \dots, f_θ be the dual basis; then $h \cdot f_i = \chi_i^{-1}(h) f_i$, $1 \leq i \leq \theta$. Set $\underline{\partial}_i = \partial_{f_i}^L$. Then

$$\underline{\partial}_i(bc) = (\chi_i \cdot b) \underline{\partial}_i(c) + \underline{\partial}_i(b) c, \quad b, c \in \mathcal{B}(W).$$

Proposition 2.3. *Let $W \in {}^H_H\mathcal{YD}$ be finite-dimensional.*

(1). *The map*

$$\Psi^L : \mathcal{B}(W) \otimes \mathcal{B}(W^*) \rightarrow \mathcal{D}^L(W)$$

is a linear isomorphism.

(2). *The map $\Theta : T(W \oplus W^*) \rightarrow \mathcal{D}^L(W)$, $(v, f) \mapsto L_v \circ \partial_f^L$, $v \in W$, $f \in W^*$, induces an algebra isomorphism $\vartheta : T(W \oplus W^*)/I \rightarrow \mathcal{D}^L(W)$, where I is the two-sided ideal generated by*

- (i) *the relations of $\mathcal{B}(W)$,*
- (ii) *the relations of $\mathcal{B}(W^*)$,*
- (iii) *the relations*

$$(2.8) \quad fv = v_{(0)} \mathcal{S}^{-1}(v_{(-1)}) \cdot f + \partial_f^L(v), \quad v \in W, f \in W^*.$$

If $v \in W$, $f \in W^*$, then (2.8) implies that

$$(2.9) \quad vf = (v_{(-1)} \cdot f) v_{(0)} - \partial_{v_{(-1)} \cdot f}^L(v_{(0)}).$$

Proof. By what was already said, Θ induces ϑ and this is surjective. Indeed, (2.6) says more generally that

$$(2.10) \quad fx = x_{(0)} \mathcal{S}^{-1}(x_{(-1)}) \cdot f + \partial_f^L(x), \quad x \in \mathcal{B}(W), f \in W^*.$$

Clearly, the inclusions of W and W^* induce algebra maps $j_W : \mathcal{B}(W) \rightarrow T(W \oplus W^*)/I$, $j_{W^*} : \mathcal{B}(W^*) \rightarrow T(W \oplus W^*)/I$. Let μ be the multiplication of $T(W \oplus W^*)/I$. Then (2.8) guarantees that $\mu \circ (j_W \otimes j_{W^*})$ is surjective. But the following diagram commutes:

$$\begin{array}{ccc} \mathcal{B}(W) \otimes \mathcal{B}(W^*) & \xrightarrow{\mu \circ (j_W \otimes j_{W^*})} & T(W \oplus W^*)/I \\ & \searrow \Psi^L & \swarrow \vartheta \\ & \mathcal{D}^L(W) & \end{array}$$

and Ψ^L is a linear isomorphism by Lemma 2.1 (ii). Thus $\mu \circ (j_W \otimes j_{W^*})$ and ϑ are isomorphisms. \square

Corollary 2.4. *Let $W \in {}^H_H\mathcal{YD}$ be finite-dimensional. Then $\mathcal{D}^L(W)$ is an algebra in ${}^H_H\mathcal{YD}$.*

Proof. Straightforward. \square

Definition 2.5. Let $\mathcal{B}(W) \# \mathcal{B}(W^*)$ denote the vector space $\mathcal{B}(W) \otimes \mathcal{B}(W^*)$ with the multiplication transported along the isomorphism $\Psi^L : \mathcal{B}(W) \otimes \mathcal{B}(W^*) \rightarrow \mathcal{D}^L(W)$. Thus

$$(2.11) \quad \xi b = \langle \xi^{(2)}, b^{(1)} \rangle (b^{(2)})_{(0)} \# \mathcal{S}^{-1}((b^{(2)})_{(-1)}) \cdot \xi^{(1)}$$

for $b \in \mathcal{B}(W)$ and $\xi \in \mathcal{B}(W^*)$ by (2.2). The multiplication map $\mu : \mathcal{B}(W^*) \otimes \mathcal{B}(W) \rightarrow \mathcal{B}(W) \# \mathcal{B}(W^*)$ is an isomorphism, with inverse map μ^- given by

$$(2.12) \quad \mu^- : b\xi \mapsto \langle \mathcal{S}_{\mathcal{B}(W^*)}((b_{(-1)} \cdot \xi)^{(2)}), b_{(0)}^{(1)} \rangle (b_{(-1)} \cdot \xi)^{(1)} \otimes b_{(0)}^{(2)}$$

for $b \in \mathcal{B}(W)$, $\xi \in \mathcal{B}(W^*)$. Note that $\Psi^L : \mathcal{B}(W) \# \mathcal{B}(W^*) \rightarrow \mathcal{D}^L(W)$ is an isomorphism in ${}^H_H\mathcal{YD}$, where H acts and coacts diagonally on $\mathcal{B}(W) \# \mathcal{B}(W^*)$, see Corollary 2.4.

Remark 2.6. Alternatively to the above construction, the algebra $\mathcal{D}^L(W)$ can be obtained as the subalgebra $\mathcal{B}(W) \# \mathcal{B}(W^*)$ of the Heisenberg double $\mathcal{A}(W) \# \mathcal{A}(W)^\circ$. Here for any Hopf algebra A the Heisenberg double $A \# A^\circ$ is the smash product algebra corresponding to the left action of the Hopf dual A° , see our convention in Remark 1.1, given by the left A -coaction on A via Δ . The embedding of $\mathcal{B}(W) \otimes \mathcal{B}(W^*)$ into $\mathcal{A}(W) \# \mathcal{A}(W)^\circ$ is given by the inclusion of $\mathcal{B}(W)$ and the map

$$\mathcal{B}(W^*) \ni f \mapsto \langle f, \cdot \rangle \otimes \varepsilon \in (\mathcal{B}(W) \# H)^\circ = \mathcal{A}(W)^\circ.$$

One can check that $\mathcal{B}(W) \otimes \mathcal{B}(W^*) \subset \mathcal{A}(W) \# \mathcal{A}(W)^\circ$ is a subalgebra and that this algebra structure on $\mathcal{B}(W) \otimes \mathcal{B}(W^*)$ coincides with $\mathcal{B}(W) \# \mathcal{B}(W^*)$ as in Definition 2.5. Further, the restriction of the map in Remark 1.8 (ii) coincides with the map in (2.12). These facts will not be used in the sequel.

Remark 2.7. Let K be a subalgebra of $\mathcal{B}(W)$ and \mathfrak{K} be a braided Hopf subalgebra of $\mathcal{B}(W^*)$ such that

- K is an H -subcomodule,
- \mathfrak{K} is an H -submodule,
- $\partial_\xi^L(b) = \langle \xi, b^{(1)} \rangle b^{(2)} \in K$ for all $b \in K$, $\xi \in \mathfrak{K}$.

Then $K \otimes \mathfrak{K}$ is a subalgebra of $\mathcal{B}(W) \# \mathcal{B}(W^*)$, denoted by $K \# \mathfrak{K}$. Again, the multiplication map $\mu : \mathfrak{K} \otimes K \rightarrow K \# \mathfrak{K}$ is an isomorphism. If $K \subset \mathcal{B}(W)$ and $\mathfrak{K} \subset \mathcal{B}(W^*)$ are subobjects in ${}^H_H\mathcal{YD}$ then $K \# \mathfrak{K}$ is a subalgebra of $\mathcal{B}(W) \# \mathcal{B}(W^*)$ in ${}^H_H\mathcal{YD}$.

Remark 2.8. Let Γ be an abelian group. Assume that $W = \bigoplus_{\gamma \in \Gamma} V_\gamma$ is a finite-dimensional Γ -graded Yetter-Drinfeld module; $W^* \simeq \bigoplus_{\gamma \in \Gamma} V_\gamma^*$ becomes a Γ -graded Yetter-Drinfeld module with $\deg V_\gamma^* = -\gamma$. Then $\mathcal{B}(W)$, $\mathcal{B}(W) \# \mathcal{B}(W^*)$, and $\mathcal{D}^L(W)$ are Γ -graded algebras.

Proof. The tensor algebras $T(W)$ and $T(W \oplus W^*)$ inherit the Γ -grading of Yetter-Drinfeld modules in the usual way: $\deg(V_{\gamma_1} \otimes \dots \otimes V_{\gamma_s}) = \gamma_1 + \dots + \gamma_s$. By definition, the braiding c preserves homogeneous components; thus $\mathcal{B}(W)$

inherits the grading. Now the relations (2.8) are also homogeneous, hence $\mathcal{B}(W) \# \mathcal{B}(W^*)$ and $\mathcal{D}^L(W)$ are Γ -graded algebras. \square

2.2. Braided derivations. We next give a characterization of Nichols algebras in terms of quantum differential operators suitable for our later purposes. Recall that the kernel of the counit of a bialgebra B is denoted by B^+ .

First, let B be a braided bialgebra and consider B^{cop} as in Remark 1.9. We write $\Delta(x) = x^{[1]} \otimes x^{[2]}$ to distinguish from the previous coproduct. Thus $\Delta(xy) = x^{[1]} y^{[1]}_{(0)} \otimes (\mathcal{S}^{-1}(y^{[1]}_{(-1)}) \cdot x^{[2]}) y^{[2]}$, for $x, y \in B$, cf. (1.9). Let $\xi \in B^*$ and let $\overline{\partial}_\xi^L \in \text{End } B$ be ∂_ξ^L for this bialgebra, that is

$$(2.13) \quad \overline{\partial}_\xi^L(x) = \langle \xi, x^{[1]} \rangle x^{[2]}.$$

Then

$$(2.14) \quad \overline{\partial}_\xi^L(xy) = (\xi^{[1]}_{(-1)} \cdot \overline{\partial}_{\xi^{[2]}}^L(x)) \overline{\partial}_{\xi^{[1]}_{(0)}}^L(y).$$

Indeed,

$$\begin{aligned} \overline{\partial}_\xi^L(xy) &= \langle \xi, x^{[1]} y^{[1]}_{(0)} \rangle (\mathcal{S}^{-1}(y^{[1]}_{(-1)}) \cdot x^{[2]}) y^{[2]} \\ &= \langle \xi^{[2]}, x^{[1]} \rangle \mathcal{S}^{-1}(y^{[1]}_{(-1)}) \cdot x^{[2]} \langle \xi^{[1]}, y^{[1]}_{(0)} \rangle y^{[2]} \\ &\stackrel{(1.11)}{=} \langle \xi^{[2]}, x^{[1]} \rangle \xi^{[1]}_{(-1)} \cdot x^{[2]} \langle \xi^{[1]}_{(0)}, y^{[1]} \rangle y^{[2]} \\ &= (\xi^{[1]}_{(-1)} \cdot \overline{\partial}_{\xi^{[2]}}^L(x)) \overline{\partial}_{\xi^{[1]}_{(0)}}^L(y). \end{aligned}$$

If $\xi \in \mathcal{P}(B) = \mathcal{P}(B^{\text{cop}})$, then

$$(2.15) \quad \overline{\partial}_\xi^L(xy) = (\xi_{(-1)} \cdot x) \overline{\partial}_{\xi_{(0)}}^L(y) + \overline{\partial}_\xi^L(x) y.$$

Part (i) of the following theorem is well-known, but part (ii) seems to be new.

Theorem 2.9. *Let $W \in {}^H_H\mathcal{YD}$ be finite-dimensional. Let $I \subset T(W)^+$ be a 2-sided ideal, stable under the action of H . Let $R = T(W)/I$ and let $\pi : T(W) \rightarrow R$ be the canonical projection.*

- (i) *Assume that I is a homogeneous Hopf ideal, so that R is a graded braided Hopf algebra quotient of $T(W)$, and that $I \cap W = 0$. Then for any $f \in W^*$ there exists a map $d_f \in \text{End } R$ such that for all $x, y \in R, v \in W$,*

$$(2.16) \quad d_f(xy) = (f_{(-1)} \cdot x) d_{f_{(0)}}(y) + d_f(x) y,$$

$$(2.17) \quad d_f(\pi(v)) = \langle f, v \rangle.$$

- (ii) Conversely, assume that for any $f \in W^*$ there exists a map $d_f \in \text{End } R$ such that (2.16) and (2.17) hold. Then $I \subseteq \mathfrak{J}(W)$, that is, there exists a unique surjective algebra map $\Omega : R \rightarrow \mathcal{B}(W)$ such that $\Omega(\pi(w)) = w$ for all $w \in W$. Moreover

$$(2.18) \quad \Omega d_f = \overline{\partial}_f^L \Omega.$$

Proof. (i). We have $R = \oplus_{n \geq 0} R^n$ with $R^1 \simeq W$; we identify W^* with a subspace of R^* , see Subsection 1.1. Hence, if $f \in W^*$, then $d_f := \overline{\partial}_f^L$ satisfies (2.16) by (2.15).

(ii). We apply (i) to $I = 0$; set $D_f \in \text{End } T(W)$, $D_f = \overline{\partial}_f^L$ for $f \in W^*$. Note that (2.17) implies that π restricted to W is injective. We claim that $d_f \pi = \pi D_f$, that is, the following diagram commutes:

$$(2.19) \quad \begin{array}{ccc} T(W) & \xrightarrow{D_f} & T(W) \\ \pi \downarrow & & \downarrow \pi \\ R & \xrightarrow{d_f} & R. \end{array}$$

For, let $\delta_f = d_f \pi$, $\tilde{\delta}_f = \pi D_f : T(W) \rightarrow R$, and let $x, y \in T(W)$. Then

$$\begin{aligned} d_f(\pi(xy)) &= (f_{(-1)} \cdot \pi(x)) d_{f_{(0)}}(\pi(y)) + d_f(\pi(x)) \pi(y); \\ \pi D_f(xy) &= \pi((f_{(-1)} \cdot x) D_{f_{(0)}}(y) + D_f(x) y) \\ &= (f_{(-1)} \cdot \pi(x)) \pi D_{f_{(0)}}(y) + \pi D_f(x) \pi(y), \end{aligned}$$

by the hypothesis on I . Also $d_f(\pi(v)) = \langle f, v \rangle = \pi D_f(v)$ for $v \in W$. Thus the set of all $x \in T(W)$ such that $\delta_f(x) = \tilde{\delta}_f(x)$ is a subalgebra that contains W ; hence $d_f \pi = \pi D_f$. (This shows that such a map d_f is unique when it exists; hence $f \mapsto d_f$ is linear in f). In other words, $D_f(I) \subset I$. Let $\langle, \rangle : T(W^*) \times T(W) \rightarrow \mathbb{k}$ be the bilinear form defined by (1.35) and (1.36), but with respect to c^{-1} . We know that $\mathfrak{J}(W, c^{-1})$ is the (right) radical of this form, and $\mathfrak{J}(W, c^{-1}) = \mathfrak{J}(W, c)$ by Lemma 1.11; so we need to show that $\langle T(W^*), I \rangle = 0$, or equivalently that $\langle T^n(W^*), I \rangle = 0$ for all $n \geq 0$. If $n = 0$, then this is clear as $\varepsilon(I) = 0$. If $n = 1$, $f \in W^*$ and $x \in I$, then $\langle f, x \rangle = \varepsilon(\langle f, x^{[1]} \rangle x^{[2]}) = \varepsilon(D_f(x)) \in \varepsilon(I) = 0$. If $n > 1$, $g \in T^{n-1}(W^*)$, $f \in W^*$ and $x \in I$, then

$$\langle gf, x \rangle = \langle f, x^{[1]} \rangle \langle g, x^{[2]} \rangle = \langle g, D_f(x) \rangle \in \langle g, D_f(I) \rangle \subset \langle g, I \rangle = 0.$$

In the following diagram, the big and upper squares commute by (i) and (2.19), respectively:

$$\begin{array}{ccc}
 T(W) & \xrightarrow{D_f} & T(W) \\
 \pi \downarrow & & \downarrow \pi \\
 R & \xrightarrow{d_f} & R \\
 \Omega \downarrow & & \downarrow \Omega \\
 \mathcal{B}(W) & \xrightarrow{\bar{\partial}_f^L} & \mathcal{B}(W)
 \end{array}
 \begin{array}{c}
 p \swarrow \quad \searrow p \\
 \end{array}$$

Hence $\bar{\partial}_f^L \Omega \pi = \bar{\partial}_f^L p = p D_f = \Omega \pi D_f = \Omega d_f \pi$, and since π is surjective, $\bar{\partial}_f^L \Omega = \Omega d_f$. \square

There are other versions of this theorem. Taking (2.4) or (2.5) into consideration, we have similar results replacing the requirement (2.16) by either of the following:

$$(2.20) \quad d_f(xy) = x_{(0)} d_{S^{-1}(x_{(-1)}) \cdot f}(y) + d_f(x)y,$$

$$(2.21) \quad d_f(xy) = x d_f(y) + d_{f_{(0)}}(x) S(f_{(-1)}) \cdot y,$$

where $x, y \in R$, $f \in W^*$. The proof goes exactly as for Theorem 2.9.

The results in Theorem 2.9 motivate the following definition.

Definition 2.10. Let $M \in {}^H\mathcal{M}$, R an algebra, T an H -module algebra, $\wp : R \rightarrow T$ an algebra map, and let $d : M \rightarrow \text{Hom}(R, T)$ be a linear map, denoted by $f \mapsto d_f$. Following [Ma93] we say that d is a *family of braided derivations* if for all $x, y \in R$, $f \in M$,

$$(2.22) \quad d_f(xy) = (f_{(-1)} \cdot \wp(x)) d_{f_{(0)}}(y) + d_f(x) \wp(y).$$

We are mostly concerned with the case when $R = T$ and $\wp = \text{id}$. In this case we say that d is a *family of braided derivations of R* .

Definition 2.11. Let $W \in {}^H_H\mathcal{YD}$. The family $d^W : W^* \rightarrow \text{End } \mathcal{B}(W)$ of braided derivations of $\mathcal{B}(W)$ with $d_f^W(w) = \langle f, w \rangle$ for all $f \in W^*$ and $w \in W$, see Theorem 2.9 (i), is called the *canonical family of braided derivations* of $\mathcal{B}(W)$.

Our next goal is to develop basic properties of families of braided derivations which will be useful in the sequel.

Lemma 2.12. Let $M \in {}^H\mathcal{M}$, V a vector space, T an H -module algebra, and $\wp : V \rightarrow T$ a linear map. Then any family of braided derivations $d : M \rightarrow \text{Hom}(T(V), T)$ determines a linear map $d^1 : M \rightarrow \text{Hom}(V, T)$ by

letting $d_f^1 = d_f|_V$, $f \in M$. Conversely, any linear map $d^1 : M \rightarrow \text{Hom}(V, T)$ gives rise to a unique family of braided derivations $d : M \rightarrow \text{Hom}(T(V), T)$, where $d_f|_V = d_f^1$, $f \in M$.

Proof. If d is a family of braided derivations, then linearity of d gives that $d^1 : M \rightarrow \text{Hom}(V, T)$ is a linear map. On the other hand, if V , $\wp : V \rightarrow T$, and $d^1 : M \rightarrow \text{Hom}(V, T)$ are given, then \wp extends uniquely to an algebra map $\wp : T(V) \rightarrow T$, and the formula

$$d_f(v_1 v_2 \cdots v_n) = \sum_{i=1}^n (f_{(1-i)} \cdot \wp(v_1)) (f_{(2-i)} \cdot \wp(v_2)) \cdots (f_{(1)} \cdot \wp(v_{i-1})) \\ \times d_{f_{(0)}}(v_i) \wp(v_{i+1}) \cdots \wp(v_n),$$

where $v_j \in V$ for all $j = 1, \dots, n$, defines a family of braided derivations $d : M \rightarrow \text{Hom}(T(V), T)$ for M , \wp , $T(V)$, and T . The uniqueness of d as a family of braided derivations follows from (2.22) and the fact that V generates the algebra $T(V)$. \square

Lemma 2.13. *Let $M \in {}^H\mathcal{M}$, V a vector space, T an H -module algebra, $\wp : T(V) \rightarrow T$ an algebra map, and $d : M \rightarrow \text{Hom}(T(V), T)$ a family of braided derivations. Let $I \subset T(V)$ be an ideal with $\wp(I) = 0$. Assume that I is generated by a subset $J \subset I$, and define $R = T(V)/I$. The following are equivalent.*

- (i) *d induces a family of braided derivations $d^R : M \rightarrow \text{Hom}(R, T)$ by letting $d_f^R(x + I) := d_f(x)$ for $x \in T(V)$, $f \in M$,*
- (ii) *$d_f(I) = 0$ for all $f \in M$,*
- (iii) *$d_f(x) = 0$ for all $f \in M$ and all generators $x \in J$.*

Proof. The implications (i) \Rightarrow (ii) and (ii) \Rightarrow (iii) are trivial. By (2.22), the linearity of d_f , and since $\wp(I) = 0$, one obtains (iii) \Rightarrow (ii). Finally, since $d^R : M \rightarrow \text{Hom}(R, T)$ is a well-defined linear map, for the implication (ii) \Rightarrow (i) it is sufficient to check (2.22). The latter holds since I is an ideal and $\wp(I) = 0$. \square

For the next theorem we need a compatibility relation between the maps ∂_g^L and ad_c .

Lemma 2.14. *Let $W \in {}^H_H\mathcal{YD}$, $w \in \mathcal{B}(W)$, $x \in W$, and $g \in W^*$. Then*

$$\partial_{w_{(-1)} \cdot g}^L(\text{ad}_c(w_{(0)})(x)) = \partial_{w_{(-1)} \cdot g}^L(w_{(0)})x \\ - (w_{(-2)} \cdot x_{(0)}) \partial_{w_{(-1)} \cdot \mathcal{S}^{-1}(x_{(-1)}) \cdot g}^L(w_{(0)}).$$

Proof. The definition of ad_c and (2.4) imply that

$$\begin{aligned}
\partial_{w_{(-1)} \cdot g}^L(\text{ad}_c(w_{(0)})(x)) &= \partial_{w_{(-1)} \cdot g}^L(w_{(0)}x) - \partial_{w_{(-2)} \cdot g}^L((w_{(-1)} \cdot x)w_{(0)}) \\
&= \partial_{w_{(-1)} \cdot g}^L(w_{(0)}x) - \partial_{w_{(-2)} \cdot g}^L(w_{(-1)} \cdot x)w_{(0)} + w_{(0)}\partial_{S^{-1}(w_{(-1)})w_{(-2)} \cdot g}^L(x) \\
&\quad - (w_{(-2)} \cdot x_{(0)})\partial_{S^{-1}(w_{(-3)}x_{(-1)}S(w_{(-1)}))w_{(-4)} \cdot g}^L(w_{(0)}) \\
&= \partial_{w_{(-1)} \cdot g}^L(w_{(0)}x) - \langle w_{(-2)} \cdot g, w_{(-1)} \cdot x \rangle w_{(0)} \\
&\quad + w \langle g, x \rangle - (w_{(-2)} \cdot x_{(0)})\partial_{w_{(-1)}S^{-1}(x_{(-1)}) \cdot g}^L(w_{(0)}).
\end{aligned}$$

The claim of the lemma now follows from (1.10). \square

We now show a very general way of constructing a family of braided derivations of $\mathcal{B}(W) \# \mathcal{B}(W^*)$. This will be crucial in the proof of Theorem 3.12, but it may be of independent interest. Recall the notion of canonical family of braided derivations d^W , see Definition 2.11.

Theorem 2.15. *Let $W \in {}^H_H\mathcal{YD}$ be finite-dimensional. For all $w \in W$ and $f = \phi_W(w) \in W^{**}$, see (1.15), define $d_f \in \text{End}(\mathcal{B}(W) \# \mathcal{B}(W^*))$ by*

$$(2.23) \quad d_f(x \# g) = -\text{ad}_c w(x) \# g + (f_{(-1)} \cdot x) \# d_{f_{(0)}}^{W^*}(g)$$

for all $x \in \mathcal{B}(W)$ and $g \in \mathcal{B}(W^*)$. Then $d : W^{**} \rightarrow \text{End}(\mathcal{B}(W) \# \mathcal{B}(W^*))$ is a family of braided derivations of $\mathcal{B}(W) \# \mathcal{B}(W^*)$.

Proof. By Proposition 2.3 and Definition 2.5 there exists a unique algebra map $\wp : T(W \oplus W^*) \rightarrow \mathcal{B}(W) \# \mathcal{B}(W^*)$ with $\wp(w) = w$, $\wp(g) = g$ for all $w \in W$, $g \in W^*$. Let $d' : W^{**} \rightarrow \text{Hom}(T(W \oplus W^*), \mathcal{B}(W) \# \mathcal{B}(W^*))$ be the unique family of braided derivations with this \wp and with

$$d'_f(x) = -\text{ad}_c(\psi_W(f))(x), \quad d'_f(g) = \langle f, g \rangle,$$

where $f \in W^{**}$, $x \in W$, and $g \in W^*$, see Lemma 2.12. We are going to use the implication (iii) \Rightarrow (i) in Lemma 2.13 to show that d' induces the family d of braided derivations of $\mathcal{B}(W) \# \mathcal{B}(W^*)$. Indeed, one has $d_f(z) = d'_f(z)$ for $z \in W \oplus W^*$. Further, for $w = \psi_W(f)$ the map $-\text{ad}_c w \in \text{End } \mathcal{B}(W)$ satisfies the relation

$$\begin{aligned}
-\text{ad}_c w(xy) &= -wxy + (w_{(-1)} \cdot (xy))w_{(0)} \\
&= -wxy + (w_{(-1)} \cdot x)w_{(0)}y - (w_{(-1)} \cdot x)w_{(0)}y + (w_{(-2)} \cdot x)(w_{(-1)} \cdot y)w_{(0)} \\
&= -\text{ad}_c w(x)y - (w_{(-1)} \cdot x) \text{ad}_c(w_{(0)})(y).
\end{aligned}$$

Thus, since ψ_W is an H -comodule map, the restriction of d'_f to $T(W)$ coincides with $-\text{ad}_c w \circ \pi_W$, where $\pi_W : T(W) \rightarrow \mathcal{B}(W)$ is the canonical map. Moreover, the restriction of d'_f to $T(W^*)$ is precisely $d_f^{W^*} \circ \pi_{W^*}$. It remains

to show that d' induces a family of braided derivations of $\mathcal{B}(W) \# \mathcal{B}(W^*)$. By the previous claims the latter family then has to coincide with the family d of linear maps d_f , where $f \in W^{**}$, and hence d is a family of braided derivations.

To see that d' induces a family of braided derivations of $\mathcal{B}(W) \# \mathcal{B}(W^*)$, we have to check that d'_f vanishes on the generators (i)–(iii) in Prop. 2.3. Since the restriction of d'_f to $T(W)$ coincides with $-\text{ad}_c(\psi(f)) \circ \pi_W$, one gets $d'_f(z) = 0$ for all $z \in \mathfrak{J}(W)$. Similarly one has $d'_f(z) = d_f(\pi_{W^*}(z))$ for all $z \in \mathfrak{J}(W^*)$. Thus it suffices to check that

$$(2.24) \quad d'_f(g \otimes x - x_{(0)} \otimes (\mathcal{S}^{-1}(x_{(-1)}) \cdot g) - \partial_g^L(x)) = 0$$

for all $x \in W$, $g \in W^*$, and $f \in W^{**}$. Note that $d'_f(\partial_g^L(x)) = 0$ since $\partial_g^L(x) \in \mathbb{k}$. Let now $x, w \in W$, $g \in W^*$, and $f = \phi_W(w) \in W^{**}$. By definition of d'_f one gets

$$\begin{aligned} d'_f(g \otimes x - x_{(0)} \otimes (\mathcal{S}^{-1}(x_{(-1)}) \cdot g)) &= d'_f(g)x + (f_{(-1)} \cdot g)d'_{f_{(0)}}(x) \\ &\quad - d'_f(x_{(0)})(\mathcal{S}^{-1}(x_{(-1)}) \cdot g) - (f_{(-1)} \cdot x_{(0)})d'_{f_{(0)}}(\mathcal{S}^{-1}(x_{(-1)}) \cdot g) \\ &= \langle f, g \rangle x - (w_{(-1)} \cdot g) \text{ad}_c(w_{(0)})(x) + \text{ad}_c w(x_{(0)})(\mathcal{S}^{-1}(x_{(-1)}) \cdot g) \\ &\quad - (w_{(-1)} \cdot x_{(0)}) \langle \phi_W(w_{(0)}), \mathcal{S}^{-1}(x_{(-1)}) \cdot g \rangle. \end{aligned}$$

Now (2.10) and Lemma 2.14 allow to simplify this expression further:

$$\begin{aligned} &= \langle f, g \rangle x - \text{ad}_c(w_{(0)})(x_{(0)})(\mathcal{S}^{-1}(w_{(-1)}x_{(-1)})w_{(-2)} \cdot g) \\ &\quad - \partial_{w_{(-1)} \cdot g}^L(\text{ad}_c(w_{(0)})(x)) + \text{ad}_c w(x_{(0)})(\mathcal{S}^{-1}(x_{(-1)}) \cdot g) \\ &\quad - (w_{(-1)} \cdot x_{(0)}) \langle \phi_W(w_{(0)}), \mathcal{S}^{-1}(x_{(-1)}) \cdot g \rangle \\ &= \langle f, g \rangle x - \langle w_{(-1)} \cdot g, w_{(0)} \rangle x + (w_{(-2)} \cdot x_{(0)}) \langle w_{(-1)} \mathcal{S}^{-1}(x_{(-1)}) \cdot g, w_{(0)} \rangle \\ &\quad - (w_{(-1)} \cdot x_{(0)}) \langle \phi_W(w_{(0)}), \mathcal{S}^{-1}(x_{(-1)}) \cdot g \rangle. \end{aligned}$$

Using the relation $f = \phi_W(w)$ and (1.16) twice, the latter expression becomes zero. This proves (2.24). \square

3. REFLECTIONS OF NICHOLS ALGEBRAS

This section is devoted to the construction of “reflections”, see (3.16). Based on them we introduce and study new invariants of Nichols algebras in ${}^H_H\mathcal{YD}$, see Definition 3.19. Then we discuss the particular class of standard semisimple Yetter-Drinfeld modules.

3.1. Braided Hopf algebras with projection. We begin by considering a commutative diagram of braided Hopf algebras in ${}^H_H\mathcal{YD}$:

$$\begin{array}{ccc} & & R \\ & \swarrow \iota & \parallel \\ S & \xrightarrow{\pi_R} & R. \end{array}$$

Here and below we use subscripts to distinguish between the various projections, coactions, etc. By bosonization, we get a commutative diagram of Hopf algebras:

$$\begin{array}{ccc} & & R\#H \\ & \swarrow \iota & \parallel \\ S\#H & \xrightarrow{\pi_{R\#H}} & R\#H \\ & \searrow \pi_{H,S} & \downarrow \pi_{H,R} \\ & & H. \end{array}$$

Clearly, the projections $\pi_{H,R} : R\#H \rightarrow H$ and $\pi_{H,S} : S\#H \rightarrow H$ satisfy

$$(3.1) \quad \pi_{H,R}\pi_{R\#H} = \pi_{H,S}.$$

We propose to study this situation through the subalgebra of coinvariants

$$(3.2) \quad K := (S\#H)^{\text{co } R\#H}.$$

We collect some basic properties of K .

Lemma 3.1. (i) K is a braided Hopf algebra in ${}^{R\#H}_{R\#H}\mathcal{YD}$ and the multiplication induces an isomorphism

$$K\#(R\#H) \simeq S\#H.$$

(ii) $K = S^{\text{co } R} = \{x \in S \mid x^{(1)} \otimes \pi_R(x^{(2)}) = x \otimes 1\}$ is a subalgebra of S and the multiplication induces an algebra isomorphism

$$K\#R \simeq S, \quad \text{cf. Remark 1.8.}$$

(iii) K is a Yetter-Drinfeld submodule over H of S and

$$(3.3) \quad \delta_H(x) = (\pi_{H,R} \otimes \text{id})\delta_{R\#H}(x), \quad x \in K.$$

(iv) $\mathcal{S}_S(K)$ is a subalgebra and Yetter-Drinfeld submodule over H of S .

Proof. (i). By the general theory of biproducts.

(ii). Let $x \in K$. By (3.1), $x_{(1)} \otimes \pi_{H,S}(x_{(2)}) = x_{(1)} \otimes \pi_{H,R}\pi_{R\#H}(x_{(2)}) = x \otimes 1$; hence $x \in S$. Now,

$$x \otimes 1 = x_{(1)} \otimes \pi_{R\#H}(x_{(2)}) = x^{(1)}(x^{(2)})_{(-1)} \otimes \pi_R((x^{(2)})_{(0)})$$

by (1.23). Applying the H -coaction to the second tensorand and then $(\mu_S \otimes \text{id})(\text{id} \otimes \mathcal{S} \otimes \text{id})$, we get

$$x \otimes 1 = x^{(1)}(x^{(2)})_{(-2)} \mathcal{S}((x^{(2)})_{(-1)}) \otimes \pi_R((x^{(2)})_{(0)}) = x^{(1)} \otimes \pi_R(x^{(2)}),$$

since π_R is H -colinear. Thus $x \in S^{\text{co} R}$.

Conversely, let $x \in S^{\text{co} R}$. Applying the H -coaction to the second tensorand of the equality $x^{(1)} \otimes \pi_R(x^{(2)}) = x \otimes 1$, and since π_R is H -colinear, we get $x \otimes 1 = x^{(1)}(x^{(2)})_{(-1)} \otimes \pi_R((x^{(2)})_{(0)}) = x_{(1)} \otimes \pi_{R \# H}(x_{(2)})$. Hence $x \in K$. The multiplication gives rise to an isomorphism because of the analogous fact in (i).

(iii). Clearly, K is an H -submodule of S . From (1.19) and (3.1) we get (3.3). Thus K is also an H -subcomodule, and *a fortiori* a Yetter-Drinfeld submodule, of S .

(iv) follows from (iii) and the properties of the antipode, *cf.* (1.25). \square

3.2. The algebra \mathcal{K} . We next work in the following general setting. Let V, W be Yetter-Drinfeld modules over H such that V is a direct summand of W in ${}^H_H\mathcal{YD}$. Or, in other words, we have a commutative diagram in ${}^H_H\mathcal{YD}$:

$$\begin{array}{ccc} & & V \\ & \swarrow \iota & \parallel \\ W & \xrightarrow{\pi} & V. \end{array}$$

Set $\tilde{V} = \ker \pi$, so that $W = V \oplus \tilde{V}$ in ${}^H_H\mathcal{YD}$. By functoriality of the Nichols algebra, we have a commutative diagram of graded Hopf algebras in ${}^H_H\mathcal{YD}$:

$$\begin{array}{ccc} & & \mathcal{B}(V) \\ & \swarrow \iota & \parallel \\ \mathcal{B}(W) & \xrightarrow{\pi_{\mathcal{B}(V)}} & \mathcal{B}(V). \end{array}$$

By bosonization, we get a commutative diagram of graded Hopf algebras:

$$(3.4) \quad \begin{array}{ccc} & & \mathcal{A}(V) = \mathcal{B}(V) \# H \\ & \swarrow \iota & \parallel \\ \mathcal{A}(W) = \mathcal{B}(W) \# H & \xrightarrow{\pi_{\mathcal{A}(V)}} & \mathcal{A}(V) = \mathcal{B}(V) \# H \\ & \searrow \pi_{H,W} & \downarrow \pi_{H,V} \\ & & H. \end{array}$$

As before, the projections $\pi_{H,V} : \mathcal{A}(V) \rightarrow H$ and $\pi_{H,W} : \mathcal{A}(W) \rightarrow H$ satisfy

$$(3.5) \quad \pi_{H,V} \pi_{\mathcal{A}(V)} = \pi_{H,W}.$$

The main actor of this section is the subalgebra of coinvariants

$$(3.6) \quad \mathcal{K} := \mathcal{A}(W)^{\text{co } \mathcal{A}(V)}.$$

Lemma 3.2. (i) \mathcal{K} is a graded braided Hopf algebra in ${}^{\mathcal{A}(V)}_{\mathcal{A}(V)}\mathcal{YD}$ and the multiplication induces an isomorphism

$$\mathcal{K} \# \mathcal{A}(V) \simeq \mathcal{A}(W).$$

(ii) $\mathcal{K} = \mathcal{B}(W)^{\text{co } \mathcal{B}(V)} = \{x \in \mathcal{B}(W) : x^{(1)} \otimes \pi_{\mathcal{B}(V)}(x^{(2)}) = x \otimes 1\}$ is a graded subalgebra of $\mathcal{B}(W)$ and the multiplication induces a homogeneous isomorphism

$$\mathcal{K} \# \mathcal{B}(V) \simeq \mathcal{B}(W).$$

(iii) \mathcal{K} is a Yetter-Drinfeld submodule over H of $\mathcal{B}(W)$ and

$$(3.7) \quad \delta_H(x) = (\pi_{H,V} \otimes \text{id}) \delta_{\mathcal{A}(V)}(x), \quad x \in \mathcal{K}.$$

(iv) $\mathcal{K} \cap W = \tilde{V} \subset \mathcal{P}(\mathcal{K})$.

Proof. (i) to (iii) are consequences of Lemma 3.1 except for statements “ \mathcal{K} is graded”, that follow since $\pi_{\mathcal{A}(V)}$ is homogeneous.

(iv). If $x \in W$, then $x^{(1)} \otimes \pi_{\mathcal{B}(V)}(x^{(2)}) = x \otimes 1 + 1 \otimes \pi_{\mathcal{B}(V)}(x)$. Hence $x \in W \cap \mathcal{K}$ if and only if $x \in \ker \pi_{\mathcal{B}(V)} \cap W = \tilde{V}$. Moreover, if $x \in \tilde{V}$, then $\vartheta_{\mathcal{K}}(x) = x$, thus $\Delta_{\mathcal{K}}(x) = x \otimes 1 + 1 \otimes x$, cf. (1.19). \square

3.3. The module L . We keep the notation of Subsection 3.2. Let U be a Yetter-Drinfeld submodule over H of \tilde{V} . We define

$$(3.8) \quad L := \text{ad } \mathcal{B}(V)(U).$$

In other words, L is the vector subspace of $\mathcal{A}(W)$ spanned by the elements

$$(3.9) \quad \text{ad}_c(x_1)(\dots(\text{ad}_c(x_m)(y))), \quad x_h \in V, 1 \leq h \leq m, y \in U,$$

for $m \geq 0$. We collect some basic properties of L .

Lemma 3.3. (i) $L = \text{ad } \mathcal{A}(V)(U)$.

(ii) $L = \bigoplus_{m \in \mathbb{N}} L^m$, where $L^m = L \cap \mathcal{B}^m(W)$; $L^1 = U$.

(iii) L is a graded Yetter-Drinfeld submodule over $\mathcal{A}(V)$ of $\mathcal{P}(\mathcal{K})$.

(iv) L is a graded Yetter-Drinfeld submodule over H of $\mathcal{P}(\mathcal{K})$.

(v) For any $x \in L$, we have

$$(3.10) \quad \Delta_{\mathcal{A}(W)}(x) \in x \otimes 1 + \mathcal{A}(V) \otimes L,$$

$$(3.11) \quad \Delta_{\mathcal{B}(W)}(x) \in x \otimes 1 + \mathcal{B}(V) \otimes L.$$

(vi) If $x \in L$ and $\pi_{\mathcal{A}(V)}(x_{(1)}) \otimes \text{pr}_1(x_{(2)}) = 0$, then $x = 0$.

(vii) If $0 \neq L'$ is an $\mathcal{A}(V)$ -subcomodule of L , then $L' \cap U \neq 0$.

Proof. (i) follows from $\text{ad } \mathcal{A}(V)(U) = \text{ad } \mathcal{B}(V) \text{ad } H(U) \subset \text{ad } \mathcal{B}(V)(U)$.

(ii). It is clear that L is a graded subspace of $\mathcal{B}(W)$ since $\mathcal{B}(V)$ is graded and U is homogeneous. Indeed, for all $m \in \mathbb{N}_0$ the space L^{m+1} is the span of the elements (3.9).

(iii). We know that $U \subset \mathcal{P}(\mathcal{K})$ by Lemma 3.2 (iv). Hence $L \subset \mathcal{P}(\mathcal{K})$ by Remark 1.6. We show that U is also an $\mathcal{A}(V)$ -subcomodule. If $y \in U$, then

$$\delta_{\mathcal{A}(V)}(y) = (\pi_{\mathcal{A}(V)} \otimes \text{id})(y \otimes 1 + y_{(-1)} \otimes y_{(0)}) = y_{(-1)} \otimes y_{(0)},$$

because $\pi_{\mathcal{A}(V)}(y) = 0$ (since $y \in \tilde{V} = \ker \pi$) and $\pi_{\mathcal{A}(V)}(y_{(-1)}) = y_{(-1)}$ (since $y_{(-1)} \in H$). By (i) and Remark 1.3 (ii), L is a Yetter-Drinfeld submodule over $\mathcal{A}(V)$ of $\mathcal{P}(\mathcal{K})$. Finally, $L^m = L \cap \mathcal{K}^m$, being the intersection of two Yetter-Drinfeld submodules, is a Yetter-Drinfeld submodule itself.

(iv) follows from (iii) and (3.7). We prove (3.10): If $x = \text{ad } z(y)$, where $z \in \mathcal{B}(V)$ and $y \in U$, then

$$\begin{aligned} \Delta_{\mathcal{A}(W)}(x) &= z_{(1)}y_{(1)}\mathcal{S}(z_{(4)}) \otimes z_{(2)}y_{(2)}\mathcal{S}(z_{(3)}) \\ &= z_{(1)}y\mathcal{S}(z_{(4)}) \otimes z_{(2)}\mathcal{S}(z_{(3)}) + z_{(1)}y_{(-1)}\mathcal{S}(z_{(3)}) \otimes \text{ad}(z_{(2)})(y_{(0)}) \\ &\in x \otimes 1 + \mathcal{A}(V) \otimes L, \end{aligned}$$

since $z_{(1)}y_{(-1)}\mathcal{S}(z_{(3)}) \in \mathcal{B}(V) \# H$ and $\text{ad}(z_{(2)})(y_{(0)}) \in L$. Here again we used that $y_{(1)} \otimes y_{(2)} = y \otimes 1 + y_{(-1)} \otimes y_{(0)}$. Now

$$\begin{aligned} \Delta_{\mathcal{B}(W)}(x) &= (\vartheta_{\mathcal{B}(W)} \otimes \text{id})\Delta_{\mathcal{A}(W)}(x) \\ &\in \vartheta_{\mathcal{B}(W)}(x) \otimes 1 + \vartheta_{\mathcal{B}(W)}(\mathcal{A}(V)) \otimes L \\ &= x \otimes 1 + \mathcal{B}(V) \otimes L. \end{aligned}$$

by (1.21), showing (3.11).

(vi). By (3.10), for some $y_i \in \mathcal{A}(V)$, $\ell_i \in L$, we have

$$\begin{aligned} 0 &= \pi_{\mathcal{A}(V)}(x_{(1)}) \otimes \text{pr}_1(x_{(2)}) = \pi_{\mathcal{A}(V)}(x) \otimes \text{pr}_1(1) + \sum_i \pi_{\mathcal{A}(V)}(y_i) \otimes \text{pr}_1(\ell_i) \\ &= \sum_i y_i \otimes \text{pr}_1(\ell_i) = x_{(1)} \otimes \text{pr}_1(x_{(2)}) = x^{(1)}(x^{(2)})_{(-1)} \otimes \text{pr}_1\left((x^{(2)})_{(0)}\right). \end{aligned}$$

As the projection pr_1 is H -colinear, we infer that

$$\begin{aligned} 0 &= x^{(1)}(x^{(2)})_{(-2)} \otimes (x^{(2)})_{(-1)} \otimes \text{pr}_1\left((x^{(2)})_{(0)}\right) \\ &\quad \text{applying } (\mu \otimes \text{id})(\text{id} \otimes \mathcal{S} \otimes \text{id}) \implies x^{(1)} \otimes \text{pr}_1(x^{(2)}) = 0. \end{aligned}$$

Since $x \in L \subset \sum_{n \geq 1} \mathcal{B}^n(W)$, we conclude that $x = 0$ by Lemma 1.12.

(vii). Let $0 \neq x \in L'$ and write $x = \sum_{1 \leq m \leq p} x(m)$ with $x(m) \in L^m$ and $y := x(p) \neq 0$. By (vi),

$$0 \neq \pi_{\mathcal{A}(V)}(y_{(1)}) \otimes \text{pr}_1(y_{(2)}) \in \mathcal{A}^{p-1}(V) \otimes \mathcal{B}^1(W).$$

Let now $F \in \text{Hom}(\mathcal{A}(V), \mathbb{k})$ such that the restriction of F to $\mathcal{A}^m(V)$ is 0 for all $m \neq p-1$. We claim that

$$F\pi_{\mathcal{A}(V)}(x_{(1)})x_{(2)} = F\pi_{\mathcal{A}(V)}(y_{(1)})\text{pr}_1(y_{(2)}).$$

Indeed,

$$\begin{aligned} F\pi_{\mathcal{A}(V)}(x_{(1)})x_{(2)} &= \sum_{1 \leq m \leq p} F\pi_{\mathcal{A}(V)}(x(m)_{(1)})x(m)_{(2)} \\ &= F\pi_{\mathcal{A}(V)}(y_{(1)})y_{(2)} \\ &= F\pi_{\mathcal{A}(V)}(y_{(1)})\text{pr}_1(y_{(2)}). \end{aligned}$$

Here the second and third equalities are clear from the assumption on F ; if $m \leq p$ then $\pi_{\mathcal{A}(V)}(x(m)_{(1)}) \otimes x(m)_{(2)} \in \oplus_{0 \leq h \leq m} \mathcal{A}^{m-h}(V) \otimes \mathcal{A}^h(W)$. Applying F we get 0 except $m = p$, $h = 1$. Choosing F appropriately, we have

$$0 \neq F\pi_{\mathcal{A}(V)}(x_{(1)})x_{(2)} = F\pi_{\mathcal{A}(V)}(y_{(1)})\text{pr}_1(y_{(2)}) \in L' \cap U.$$

□

Part (vii) of Lemma 3.3 implies some strong restrictions on the Yetter-Drinfeld submodules of L .

Proposition 3.4. *Assume that $U = U_1 \oplus \cdots \oplus U_\theta$ in ${}^H_H\mathcal{YD}$. Let $L_i = \text{ad } \mathcal{A}(V)(U_i)$. Then $L = L_1 \oplus \cdots \oplus L_\theta$ in ${}^{\mathcal{A}(V)}_{\mathcal{A}(V)}\mathcal{YD}$.*

Proof. We have to show that the sum $L_1 + \cdots + L_\theta$ is direct. Suppose that $L_i \cap (\sum_{j \neq i} L_j) \neq 0$; then $L_i \cap (\sum_{j \neq i} L_j) \cap U \neq 0$ by Lemma 3.3(vii). Note that $(\sum_{j \neq i} L_j) \cap U = (\sum_{j \neq i} L_j)^1 \cap U = \sum_{j \neq i} U_j$. Thus $L_i \cap (\sum_{j \neq i} L_j) \cap U = U_i \cap (\sum_{j \neq i} U_j) \neq 0$, a contradiction. □

Clearly, if $U' \subsetneq U$ in ${}^H_H\mathcal{YD}$, then $L' := \text{ad } \mathcal{A}(V)(U') \subsetneq L$ in ${}^{\mathcal{A}(V)}_{\mathcal{A}(V)}\mathcal{YD}$. Hence, if L is irreducible in ${}^{\mathcal{A}(V)}_{\mathcal{A}(V)}\mathcal{YD}$, then U is irreducible in ${}^H_H\mathcal{YD}$. The converse holds because of Lemma 3.3(vii).

Proposition 3.5. *If U is irreducible in ${}^H_H\mathcal{YD}$, then L is irreducible in ${}^{\mathcal{A}(V)}_{\mathcal{A}(V)}\mathcal{YD}$.*

Proof. Let $0 \neq L'$ be a subobject of L in ${}^{\mathcal{A}(V)}_{\mathcal{A}(V)}\mathcal{YD}$. Then $L' \cap U \neq 0$ by Lemma 3.3(vii). Since both L' and U are H -stable, $L' \cap U$ is an H -submodule

of U . It is an H -subcomodule of U by (3.7); thus $L' \cap U \hookrightarrow U$ in ${}^H_H\mathcal{YD}$. By the irreducibility assumption, $L' \cap U = U$, hence $L = \text{ad } \mathcal{A}(V)(U) \subseteq L'$. \square

If $U = \tilde{V}$, then we have the following property, important for our later considerations.

Proposition 3.6. *The algebra \mathcal{K} is generated by $\text{ad } \mathcal{B}(V)(\tilde{V})$.*

Proof. Let \mathcal{K}' be the subalgebra of \mathcal{K} generated by $\text{ad } \mathcal{B}(V)(\tilde{V})$ and let X be the image of $\mathcal{K}' \# \mathcal{B}(V)$ under the isomorphism $\mathcal{K} \# \mathcal{B}(V) \simeq \mathcal{B}(W)$ given by multiplication. It suffices to prove that $X = \mathcal{B}(W)$. Since $V \subset X$ and $\tilde{V} \subset X$, one gets $W \subset X$; it remains then to show that X is a subalgebra of $\mathcal{B}(W)$. For this, observe that \mathcal{K}' is stable under the adjoint action of $\mathcal{A}(V)$. Indeed, $\text{ad } x(yy') = \text{ad}(x_{(1)})(y) \text{ad}(x_{(2)})(y')$, for all $x \in \mathcal{A}(V)$, $y, y' \in \text{ad } \mathcal{B}(V)(\tilde{V})$. Hence, if $x \in V$ and $y \in \mathcal{K}'$, then $xy = \text{ad}_c x(y) + (x_{(-1)} \cdot y)x_{(0)} \in \mathcal{K}' + \mathcal{K}' \# V \subset X$. As both \mathcal{K}' and $\mathcal{B}(V)$ are subalgebras, we conclude that X is a subalgebra and the proposition follows. \square

We now introduce the following finiteness condition on U . Recall that $L = \text{ad } \mathcal{B}(V)(U)$.

(F) $L^M \neq 0$ and $L^p = 0$ for some $M \in \mathbb{N}$ and all $p > M$.

Clearly, a sufficient condition for (F) is that $L = \bigoplus_{m \in \mathbb{N}} L^m$ has finite dimension. In this case, $\dim U < \infty$ too.

If M is determined by (F), then we write

$$(3.12) \quad L^{\max} := L^M.$$

Lemma 3.7. *Assume that U satisfies Condition (F). Let Z be a Yetter-Drinfeld submodule over H of L^{\max} and $\langle Z \rangle$ the $\mathcal{B}(V)$ -subcomodule of L generated by Z .*

- (i) $\langle Z \rangle = \bigoplus_{m=1}^M \langle Z \rangle^m$, where $\langle Z \rangle^m = \langle Z \rangle \cap \mathcal{B}^m(W)$ for all m , and $\langle Z \rangle^M = Z$.
- (ii) $\langle Z \rangle$ is the $\mathcal{A}(V)$ -subcomodule of L generated by Z .
- (iii) $\langle Z \rangle$ is a graded Yetter-Drinfeld submodule over $\mathcal{A}(V)$ of L .
- (iv) $\langle Z \rangle$ is a graded Yetter-Drinfeld submodule over H of L .

Proof. By (1.6), $\langle Z \rangle$ is the vector subspace of L spanned by the elements

$$\langle f, z_{(-1)} \rangle z_{(0)}, \quad \text{where } z \in Z, f \in \mathcal{B}^n(V)^*, \quad n \geq 0,$$

where $z_{(-1)} \otimes z_{(0)} = \delta_{\mathcal{B}(V)}(z)$. Let $z \in Z$ and $f \in \mathcal{B}^n(V)^*$. We obtain that $\langle f, z_{(-1)} \rangle z_{(0)} \in \langle Z \rangle^{M-n}$, since

$$z_{(-1)} \otimes z_{(0)} = \pi_{\mathcal{B}(V)}(z^{(1)}) \otimes z^{(2)} \in \bigoplus_{m \in \mathbb{N}_0} \mathcal{B}^m(V) \otimes \mathcal{B}^{M-m}(W).$$

This proves (i). Now (ii) follows from Lemma 1.5 (iii); then (iii) follows from (ii), Assumption (F), and Remark 1.3 (i), while (iv) follows from (iii) and (3.7). \square

We can now present the first ingredient of our construction in (3.16).

Theorem 3.8. *Suppose that U is irreducible in ${}^H_H\mathcal{YD}$ and satisfies Condition (F). Then L^{\max} is irreducible in ${}^H_H\mathcal{YD}$ and L is generated by L^{\max} as a $\mathcal{B}(V)$ -comodule.*

Proof. By Proposition 3.5, L is irreducible in ${}^{\mathcal{A}(V)}_{\mathcal{A}(V)}\mathcal{YD}$. If $0 \neq Z \hookrightarrow L^{\max}$ in ${}^H_H\mathcal{YD}$, then $0 \neq \langle Z \rangle = \mathcal{B}(V)^* \cdot Z \hookrightarrow L$ in ${}^{\mathcal{A}(V)}_{\mathcal{A}(V)}\mathcal{YD}$ by Lemma 3.7 (iii). Thus $\langle Z \rangle = L$, and $Z = \langle Z \rangle^M = L^M = L^{\max}$ by Lemma 3.7 (i). \square

3.4. Reflections. Let us fix $\theta \in \mathbb{N}$ and let $\mathbb{I} = \{1, \dots, \theta\}$. Let \mathcal{C}_θ denote the class of all families

$$M = (M_1, \dots, M_\theta)$$

of finite-dimensional irreducible Yetter-Drinfeld modules $M_j \in {}^H_H\mathcal{YD}$, where $j \in \mathbb{I}$. Two families $M, M' \in \mathcal{C}_\theta$ are called *isomorphic* if M_j is isomorphic to M'_j in ${}^H_H\mathcal{YD}$ for all $j \in \mathbb{I}$. In this case we write $M \simeq M'$.

Let $(\alpha_1, \dots, \alpha_\theta)$ be the standard basis of \mathbb{Z}^θ . Let $M = (M_1, \dots, M_\theta) \in \mathcal{C}_\theta$ and

$$(3.13) \quad W = \bigoplus_{j=1}^\theta M_j.$$

Define a \mathbb{Z}^θ -grading on W by $\deg M_j = \alpha_j$ for all $j \in \mathbb{I}$. We fix $i \in \mathbb{I}$ and set

$$V = M_i, \quad \tilde{V} = \bigoplus_{j \in \mathbb{I}, j \neq i} M_j.$$

Thus, we are in the situation of Subsections 3.2 and 3.3. Let

$$(3.14) \quad L_j := \text{ad } \mathcal{B}(V)(M_j) \quad \text{for } j \in \mathbb{I} \setminus \{i\}.$$

Thus, L_j is the vector subspace of $\mathcal{B}(W)$ spanned by the elements

$$\text{ad}_c(x_1)(\dots(\text{ad}_c(x_m)(y))), \quad x_h \in M_i, \ 1 \leq h \leq m, \ y \in M_j, \ m \geq 0.$$

Recall that $\mathcal{K} = \mathcal{A}(W)^{\text{co } \mathcal{A}(V)} = \mathcal{B}(W)^{\text{co } \mathcal{B}(V)}$, see (3.6) and Lemma 3.2 (ii). Consider the \mathbb{Z}^θ -grading on the algebras $\mathcal{B}(W)$ and $\mathcal{B}(V)$ discussed in Remark 2.8, page 19. Then the algebras $\mathcal{A}(W)$ and $\mathcal{A}(V)$ are also \mathbb{Z}^θ -graded, by setting $\deg H = 0$. Since the map $\pi_{\mathcal{A}(V)}$ in (3.4) is homogeneous, the algebra \mathcal{K} inherits this grading. Then L_j is a \mathbb{Z}^θ -graded subspace of \mathcal{K} and $\text{supp } L_j \subset \alpha_j + \mathbb{N}_0 \alpha_i$. Let

$$(3.15) \quad -a_{ij}^M := \sup\{h \in \mathbb{N}_0 \mid \alpha_j + h\alpha_i \in \text{supp } L_j\}.$$

Then either $a_{ij}^M \in \mathbb{Z}_{\leq 0}$ (when $\text{supp } L_j$ is finite), or $a_{ij}^M = -\infty$. Let also $a_{ii}^M = 2$.

We introduce the following finiteness conditions for M .

(F_i) $\dim L_j$ is finite for all $j \in \mathbb{I}$, $j \neq i$,

or, equivalently,

(F'_i) $\text{supp } L_j$ is finite for all $j \in \mathbb{I}$, $j \neq i$.

Note that (F_i) means that $a_{ij}^M > -\infty$ for all $j \in \mathbb{I} \setminus \{i\}$.

Remark 3.9. It would be interesting to find an *a priori* condition guaranteeing that (F_i) holds. Obviously, if $\dim \mathcal{B}(W) < \infty$, then $\dim L_j < \infty$ for all j . Because of [R98], we believe that (F_i) holds whenever the Gelfand-Kirillov dimension of $\mathcal{B}(W)$ is finite.

Assume that M satisfies Condition (F_i) . Let $s_{i,M} \in GL(\theta, \mathbb{Z})$ and

$$(3.16) \quad \mathcal{R}_i(M) := (M'_1, \dots, M'_\theta) \in \mathcal{C}_\theta$$

be given by

$$(3.17) \quad s_{i,M}(\alpha_j) = \alpha_j - a_{ij}^M \alpha_i, \quad j \in \mathbb{I},$$

$$(3.18) \quad M'_j = \begin{cases} L_j^{\max} & \text{if } j \neq i, \\ M_i^* = V^* & \text{if } j = i. \end{cases}$$

Notice that $\mathcal{R}_i(M)$ is an object of \mathcal{C}_θ by Theorem 3.8. We say that \mathcal{R}_i is the *i-th reflection*. The linear map $s_{i,M}$ is a reflection in the sense of [B68, Ch. V, §2.2], that is, $s_{i,M}^2 = \text{id}$ and the rank of $\text{id} - s_{i,M}$ is 1.

We embed V^* into W^* via the decomposition of W in (3.13). Then

$$\mathcal{K} \# \mathcal{B}(V^*) \subset \mathcal{B}(W) \# \mathcal{B}(W^*)$$

is a subalgebra, see Definition 2.5. Further, $\mathcal{K} \# \mathcal{B}(V^*)$ is a \mathbb{Z}^θ -graded algebra in ${}^H_H \mathcal{YD}$ with $\deg x = s_{i,M}(\alpha_i) = -\alpha_i$ for all $x \in V^*$, see Remark 2.8.

Lemma 3.10. *The map $T(\mathcal{K} \oplus V^*) \rightarrow \mathcal{K} \# \mathcal{B}(V^*)$, $\mathcal{K} \oplus V^* \ni (x, f) \mapsto x \# 1 + 1 \# f$, induces an algebra isomorphism $T(\mathcal{K} \oplus V^*)/I \rightarrow \mathcal{K} \# \mathcal{B}(V^*)$, where I is the two-sided ideal generated by*

- (i) the elements $x \otimes y - xy$, where $x, y \in \mathcal{K}$, and $1_{\mathcal{K}} - 1_{T(\mathcal{K} \oplus V^*)}$,
- (ii) the relations of $\mathcal{B}(V^*)$,
- (iii) the elements

$$(3.19) \quad g \otimes x - x_{(0)} \otimes \mathcal{S}^{-1}(x_{(-1)}) \cdot g - \partial_g^L(x), \quad x \in \mathcal{K}, g \in V^*.$$

Proof. See the proof of Proposition 2.3. □

Let $W' = \oplus_{j=1}^\theta M'_j$. Then W' is contained in $\mathcal{K} \# \mathcal{B}(V^*)$ via the embeddings

$$\begin{aligned} M'_j &\subset \mathcal{K} \simeq \mathcal{K} \# \mathbb{k} \subset \mathcal{K} \# \mathcal{B}(V^*) \quad \text{for all } j \neq i, \\ M'_i &= V^* \simeq \mathbb{k} \# V^* \subset \mathcal{K} \# \mathcal{B}(V^*)^+. \end{aligned}$$

Moreover, W' inherits a \mathbb{Z}^θ -grading from $\mathcal{B}(W) \# \mathcal{B}(W^*)$: One has $\deg M'_j = s_{i,M}(\alpha_j)$ for all $j \in \mathbb{I}$.

Lemma 3.11. *The algebra $\mathcal{K} \# \mathcal{B}(V^*)$ is generated by W' .*

Proof. Let $\mathfrak{B} = \mathbb{k}\langle W' \rangle$ be the subalgebra of $\mathcal{K} \# \mathcal{B}(V^*)$ generated by W' . Since $W' \in {}^H_H\mathcal{YD}$, \mathfrak{B} is a subobject of $\mathcal{B}(W) \# \mathcal{B}(W^*)$ in ${}^H_H\mathcal{YD}$ by Corollary 2.4. Fix $j \neq i$ and pick $x \in L_j \cap \mathfrak{B}$, $f \in V^*$. Then $fx = x_{(0)} \mathcal{S}^{-1}(x_{(-1)}) \cdot f + \partial_f^L(x)$ by (2.10). Now, $L_j \cap \mathfrak{B}$ being a Yetter-Drinfeld submodule over H , this says that $\partial_f^L(x) \in \mathfrak{B}$. But

$$\partial_f^L(x) = \langle f, x^{(1)} \rangle x^{(2)} \in \langle f, x \rangle 1 + \langle f, \mathcal{B}(V) \rangle L_j \subset L_j,$$

by (3.11). This shows that $L_j \cap \mathfrak{B}$ is a $\mathcal{B}(V)$ -subcomodule of L_j ; indeed, $\langle f, x_{(-1)} \rangle x_{(0)} = \langle f, \pi_{\mathcal{B}(V)}(x^{(1)}) \rangle x^{(2)} = \langle f, x^{(1)} \rangle x^{(2)}$. We conclude that $L_j \cap \mathfrak{B} = L_j$ by Lemma 3.7 (iii) and Prop. 3.5, since $0 \neq L_j \cap \mathfrak{B} \supset L_j^{\max}$. Hence $L_j \subset \mathfrak{B}$ for $j \in \mathbb{I} \setminus \{i\}$, and Prop. 3.6 implies that $\mathcal{K} \subset \mathfrak{B}$. This proves the lemma. \square

Here is our first main result.

Theorem 3.12. *Let $M = (M_1, \dots, M_\theta) \in \mathcal{C}_\theta$ and $i \in \mathbb{I}$ such that M satisfies Condition (F_i) . Let $V = M_i$, $W = \bigoplus_{j \in \mathbb{I}} M_j$, $\mathcal{K} = \mathcal{B}(W)^{\text{co } \mathcal{B}(V)}$, $M' = \mathcal{R}_i(M)$ and $W' = \bigoplus_{j \in \mathbb{I}} M'_j$. Define a \mathbb{Z}^θ -grading on W' by $\deg x = s_{i,M}(\alpha_j)$ for all $x \in M'_j$, $j \in \mathbb{I}$.*

- (1) *The inclusion $W' \hookrightarrow \mathcal{K} \# \mathcal{B}(V^*)$ induces a \mathbb{Z}^θ -homogeneous isomorphism $\mathcal{B}(W') \simeq \mathcal{K} \# \mathcal{B}(V^*)$ of algebras and of Yetter-Drinfeld modules over H .*
- (2) *The family $\mathcal{R}_i(M)$ satisfies Condition (F_i) , and $s_{i,\mathcal{R}_i(M)} = s_{i,M}$, $\mathcal{R}_i^2(M) \simeq M$.*

We prove the theorem in several steps. The strategy of the proof is the following. First we define a surjective algebra map $\Omega : \mathcal{K} \# \mathcal{B}(V^*) \rightarrow \mathcal{B}(W')$. Then we conclude that the same construction can be performed for M' instead of M , and that (2) holds. Finally we prove that Ω is bijective. The restriction of the inverse map of Ω to W' is the given embedding of W' in $\mathcal{K} \# \mathcal{B}(V^*)$.

For the definition of Ω , see Prop. 3.14, we use the characterization of Nichols algebras in Theorem 2.9 (ii). In the next lemma we prove the existence of the required family of braided derivations.

Main Lemma 3.13. *There is a unique family $d : W'^* \rightarrow \text{End}(\mathcal{K} \# \mathcal{B}(V^*))$ of braided derivations of $\mathcal{K} \# \mathcal{B}(V^*)$ such that*

$$(3.20) \quad d_f(w') = \langle f, w' \rangle$$

for all $f \in W'^* \simeq V^{**} \oplus \bigoplus_{j \in \mathbb{I} \setminus \{i\}} (L_j^{\max})^*$, $w' \in W'$. Moreover, for all $v \in V$, $f = \phi_V(v) \in V^{**}$, and $x \in \mathcal{K}$ equation $d_f(x) = -\text{ad}_c v(x)$ holds.

Proof. The family d is unique since $\mathcal{K} \# \mathcal{B}(V^*)$ is generated by W' , see Lemma 3.11. By Definition 2.10 it is sufficient to show that

- (1) there exists a family $d : V^{**} \rightarrow \text{End}(\mathcal{K} \# \mathcal{B}(V^*))$ of braided derivations of $\mathcal{K} \# \mathcal{B}(V^*)$ such that $d_f(w') = \langle f, w' \rangle$ for all $f \in V^{**}$ and $w' \in W'$,
- (2) for all $j \in \mathbb{I} \setminus \{i\}$ there exists a family $d : (L_j^{\max})^* \rightarrow \text{End}(\mathcal{K} \# \mathcal{B}(V^*))$ of braided derivations of $\mathcal{K} \# \mathcal{B}(V^*)$ such that $d_f(w') = \langle f, w' \rangle$ for all $f \in (L_j^{\max})^*$ and $w' \in W'$.

First we prove (1). Let $d : V^{**} \rightarrow \text{End}(\mathcal{B}(W) \# \mathcal{B}(W^*))$ be the restriction to V^{**} of the family of braided derivations in Theorem 2.15. By (2.23) one gets

$$(3.21) \quad d_f(x) = -\text{ad}_c v(x) \quad \text{for all } v \in V, f = \phi_V(v) \in V^{**}, x \in \mathcal{K}.$$

Thus $d_f(\mathcal{K}) \subset \mathcal{K}$ for all $f \in V^{**}$, and $d_f(\mathcal{B}(V^*)) \subset \mathcal{B}(V^*)$ since $d_f(w') = \langle f, w' \rangle$ for all $w' \in V^*$ by (2.23). Hence d induces a family of braided derivations of $\mathcal{K} \# \mathcal{B}(V^*)$ by restriction. The relation $d_f(w') = \langle f, w' \rangle = 0$ for $w' \in L_j^{\max}$, $j \neq i$, follows from the definition of L_j^{\max} , and the second claim of the lemma holds by (3.21).

To prove (2), let $j \in \mathbb{I} \setminus \{i\}$. We first define a family $d : (L_j^{\max})^* \rightarrow \text{End}(\mathcal{K})$ of braided derivations of \mathcal{K} . Then we extend d to a family of braided derivations of $\mathcal{K} \# \mathcal{B}(V^*)$.

Recall from (3.17) that $s_{i,M}(\alpha_j) = \alpha_j - a_{ij}^M \alpha_i$. Define $d_F : \mathcal{B}(W) \rightarrow \mathcal{B}(W)$ for any $F \in \mathcal{B}(W^*)_{-s_{i,M}(\alpha_j)}$ by

$$(3.22) \quad d_F(x) := \langle F, (x^{(2)})_{(0)} \rangle \mathcal{S}^{-1}((x^{(2)})_{(-1)}) \cdot x^{(1)}, \quad x \in \mathcal{B}(W),$$

see (2.13). Then

$$(3.23) \quad d_F(x) = 0 \quad \text{if } x \in L_h, h \neq j, i, \text{ or } x \in L_j^m, m < 1 - a_{ij}^M.$$

Indeed, if $x \in L_h^m$, where $h \in \mathbb{I} \setminus \{i\}$, $m \in \mathbb{N}$, then by Lemma 3.3 (iii) and (3.11) one gets

$$\Delta(x) \in x \otimes 1 + 1 \otimes x + \sum_{0 < r < m} \mathcal{B}(W)_{r\alpha_i} \otimes \mathcal{B}(W)_{\alpha_h + (m-1-r)\alpha_i}.$$

Hence $\langle F, (x^{(2)})_{(0)} \rangle = 0$ whenever $h \neq j$ or $h = j$, $m < 1 - a_{ij}^M$. Further, if $x \in L_j^{1-a_{ij}^M}$ then

$$(3.24) \quad d_F(x) = \langle F, x \rangle \quad \text{for all } x \in L_j^{1-a_{ij}^M}.$$

We next claim that

$$(3.25) \quad d_F(xy) := d_F(x)y + (F_{(-1)} \cdot x)d_{F_{(0)}}(y) \quad \text{for all } x, y \in \mathcal{K}.$$

Let $x, y \in \mathcal{K}$. Then

$$(3.26) \quad \begin{aligned} d_F(xy) = & \langle F, (x^{(2)})_{(0)}(y^{(2)})_{(0)} \rangle \\ & \times \mathfrak{S}^{-1}((x^{(2)})_{(-1)}(y^{(2)})_{(-1)}) \cdot [x^{(1)}((x^{(2)})_{(-2)} \cdot y^{(1)})]. \end{aligned}$$

Now $\langle F, (x^{(2)})_{(0)}(y^{(2)})_{(0)} \rangle = \langle F^{(1)}, (y^{(2)})_{(0)} \rangle \langle F^{(2)}, (x^{(2)})_{(0)} \rangle$. Further,

$$\begin{aligned} \Delta(F) - F \otimes 1 - 1 \otimes F \in & \sum_{0 < r < 1 - a_{ij}^M} (\mathcal{B}(W^*)_{-r\alpha_i} \otimes \mathcal{B}(W^*)_{-s_{i,M}(\alpha_j) + r\alpha_i} \\ & + \mathcal{B}(W^*)_{-s_{i,M}(\alpha_j) + r\alpha_i} \otimes \mathcal{B}(W^*)_{-r\alpha_i}). \end{aligned}$$

Since $\mathcal{K} \subset \mathcal{B}(W)$ is a left coideal and $\langle F', \mathcal{K} \rangle = 0$ for all $F' \in \mathcal{B}(W^*)_{-r\alpha_i}$ and $r > 0$, one gets

$$\langle F, (x^{(2)})_{(0)}(y^{(2)})_{(0)} \rangle = \langle F, (y^{(2)})_{(0)} \rangle \varepsilon((x^{(2)})_{(0)}) + \varepsilon((y^{(2)})_{(0)}) \langle F, (x^{(2)})_{(0)} \rangle.$$

This means that d_F behaves in the same way as $d_{F'}$ for primitive F' , and hence (3.25) follows from (2.15).

We point out two consequences of the claim (3.25). First, this shows that $d_F(\mathcal{K}) \subset \mathcal{K}$; indeed, \mathcal{K} is generated as an algebra by L and we know already that $d_F(L) \subset \mathcal{K}$ by (3.23) and (3.24). Second, the inclusion $L_j^{1-a_{ij}^M} \subset \mathcal{B}(W)_{s_{i,M}(\alpha_j)}$ induces a projection $\pi : \mathcal{B}(W^*)_{-s_{i,M}(\alpha_j)} \rightarrow (L_j^{1-a_{ij}^M})^*$; then $d_F \in \text{End } \mathcal{K}$ depends only on $f = \pi(F)$. For, if $\pi(F) = 0$, then $d_F = 0$ on L by (3.23) and (3.24). Hence $d_F = 0$ on \mathcal{K} by (3.25). Thus we have constructed the desired family $d : (L_j^{\max})^* \rightarrow \text{End } \mathcal{K}$ of braided derivations of \mathcal{K} .

Now we extend d to a family of braided derivations of $\mathcal{K} \# \mathcal{B}(V^*)$ by letting

$$(3.27) \quad d_f(xg) = d_f(x)g, \quad x \in \mathcal{K}, g \in \mathcal{B}(V^*).$$

It is clear that $d_f(w') = \langle f, w' \rangle$ for all $f \in (L_j^{\max})^*$, $w' \in W'$. It remains to prove that

$$(3.28) \quad d_f(bc) = (f_{(-1)} \cdot b) d_{f_{(0)}}(c) + d_f(b)c$$

for all $b, c \in \mathcal{K} \# \mathcal{B}(V^*)$ and $f = \pi(F) \in (L_j^{1-a_{ij}^M})^*$. Similarly to the proof of Theorem 2.15, we use Lemma 2.13 (iii) \Rightarrow (i) and Lemma 3.10 to show that $d : (L_j^{\max})^* \rightarrow \text{End}(\mathcal{K} \# \mathcal{B}(V^*))$, given in (3.27), defines a family of braided derivations of $\mathcal{K} \# \mathcal{B}(V^*)$. Again it suffices to check that

$$(3.29) \quad d'_f(g \otimes x) = d'_f(x_{(0)} \otimes \mathcal{S}^{-1}(x_{(-1)}) \cdot g + \partial_g^L(x))$$

for all $x \in \mathcal{K}$, $g \in V^*$, where $d' : (L_j^{\max})^* \rightarrow \text{Hom}(T(\mathcal{K} \oplus V^*), \mathcal{K} \# \mathcal{B}(V^*))$ denotes the family of braided derivations induced by $d'_f|_{\mathcal{K}} = d_f$ and $d'_f|_{V^*} = 0$. The right-hand side of (3.29) is

$$\begin{aligned} & d'_f(x_{(0)} \otimes \mathcal{S}^{-1}(x_{(-1)}) \cdot g + \langle g, x^{(1)} \rangle x^{(2)}) \\ &= d_F(x_{(0)}) \mathcal{S}^{-1}(x_{(-1)}) \cdot g + d_F(\langle g, x^{(1)} \rangle x^{(2)}) \\ &= \langle F, (x^{(2)})_{(0)} \rangle (\mathcal{S}^{-1}((x^{(2)})_{(-1)}) \cdot (x^{(1)})_{(0)}) (\mathcal{S}^{-1}((x^{(1)})_{(-1)}) (x^{(2)})_{(-2)}) \cdot g \\ &\quad + \langle g, x^{(1)} \rangle \langle F, (x^{(3)})_{(0)} \rangle \mathcal{S}^{-1}((x^{(3)})_{(-1)}) \cdot x^{(2)}, \end{aligned}$$

and the left-hand side is

$$\begin{aligned} & (f_{(-1)} \cdot g) d'_{f_{(0)}}(x) + d'_f(g)x = (F_{(-1)} \cdot g) d_{F_{(0)}}(x) \\ & \stackrel{(3.22)}{=} (F_{(-1)} \cdot g) \langle F_{(0)}, (x^{(2)})_{(0)} \rangle \mathcal{S}^{-1}((x^{(2)})_{(-1)}) \cdot x^{(1)} \\ & \stackrel{(1.34)}{=} (\mathcal{S}^{-1}((x^{(2)})_{(-1)}) \cdot g) \langle F, (x^{(2)})_{(0)} \rangle \mathcal{S}^{-1}((x^{(2)})_{(-2)}) \cdot x^{(1)} \\ &= \langle F, (x^{(2)})_{(0)} \rangle \mathcal{S}^{-1}((x^{(2)})_{(-1)}) \cdot (gx^{(1)}) \\ & \stackrel{(2.10)}{=} \langle F, (x^{(2)})_{(0)} \rangle \mathcal{S}^{-1}((x^{(2)})_{(-1)}) \cdot ((x^{(1)})_{(0)} \mathcal{S}^{-1}((x^{(1)})_{(-1)}) \cdot g) \\ &\quad + \langle F, (x^{(3)})_{(0)} \rangle \mathcal{S}^{-1}((x^{(3)})_{(-1)}) \cdot \langle g, x^{(1)} \rangle x^{(2)}. \end{aligned}$$

This proves (3.29) and completes the proof of the lemma. \square

Recall the notation from Theorem 3.12.

Proposition 3.14. *There exists a unique surjective algebra map*

$$\Omega : \mathcal{K} \# \mathcal{B}(V^*) \rightarrow \mathcal{B}(W')$$

which is the identity on W' . Define a \mathbb{Z}^θ -grading on W' by $\deg x = s_{i,M}(\alpha_j)$ for all $x \in M'_j$, $j \in \mathbb{I}$. Then Ω is a \mathbb{Z}^θ -graded map in ${}^H_H\mathcal{YD}$, and for all $v \in V$, $f \in V^*$, $x \in \mathcal{K}$ the following equations hold.

$$(3.30) \quad \Omega(\partial_f^L(x)) = \text{ad}_{c^{-1}} f(\Omega(x)),$$

$$(3.31) \quad \Omega(\text{ad}_c v(x)) = -d_{\phi_V(v)}^{W'}(\Omega(x)).$$

Proof. By Lemma 3.11 there is a unique surjective algebra map $T(W') \rightarrow \mathcal{K}\#\mathcal{B}(V^*)$ which is the identity on W' . Let I be the kernel of this map. Since $\mathcal{K}\#\mathcal{B}(V^*)$ is an H -module, I is invariant under the action of H . By Main Lemma 3.13 there is a unique family $d : W'^* \rightarrow \text{End}(\mathcal{K}\#\mathcal{B}(V^*))$ of braided derivations satisfying (3.20). Thus Theorem 2.9 (ii) applies, that is, the algebra map Ω exists and is unique. By definition of the \mathbb{Z}^θ -gradings and the Yetter-Drinfeld structures, Ω is a \mathbb{Z}^θ -graded map in ${}^H_H\mathcal{YD}$.

(3.31) follows from (2.18) by using the second part of Main Lemma 3.13, Equations (2.13), (2.15), and Definition 2.11. (3.30) follows from the formulas

$$\begin{aligned} \Omega(\partial_f^L(x)) &\stackrel{(2.10)}{=} \Omega(fx - x_{(0)}(\mathcal{S}^{-1}(x_{(-1)}) \cdot f)) \\ &= f\Omega(x) - \Omega(x_{(0)})(\mathcal{S}^{-1}(x_{(-1)}) \cdot f) = \text{ad}_{c^{-1}} f(\Omega(x)). \end{aligned}$$

□

Now we are prepared to prove Theorem 3.12.

Proof of Theorem 3.12. We follow the strategy explained below Theorem 3.12. Recall that $L_j = \text{ad } \mathcal{B}(V)(M_{\alpha_j})$ and

$$(3.32) \quad \begin{aligned} L_j &= \mathbb{k}\text{-span of } \{\partial_{f_1}^L \cdots \partial_{f_n}^L(x) \mid x \in L_j^{\max} = M'_j, \\ &\quad n \geq 0, f_1, \dots, f_n \in V^*\} \end{aligned}$$

for all $j \in \mathbb{I} \setminus \{i\}$ by Theorem 3.8. Let $M' = \mathcal{R}_i(M)$ as in (3.16),

$$(3.33) \quad L'_j = \text{ad } \mathcal{B}(V^*)(M'_{\beta_j}) \subset \mathcal{B}(W'),$$

and $\Omega : \mathcal{K}\#\mathcal{B}(V^*) \rightarrow \mathcal{B}(W')$ the epimorphism in Prop. 3.14.

We first claim that

$$(3.34) \quad \tilde{\Omega}|_{L_j} : L_j \rightarrow L'_j \quad \text{is bijective,}$$

where $\tilde{\Omega} = \mathcal{S}_{\mathcal{B}(W')}\Omega$. Indeed, let $x \in L_j^{\max}$, $n \geq 0$, and $f_1, \dots, f_n \in V^*$. Then

$$\begin{aligned} \mathcal{S}_{\mathcal{B}(W')}\Omega(\partial_{f_1}^L \cdots \partial_{f_n}^L(x)) &\stackrel{(3.30)}{=} \mathcal{S}_{\mathcal{B}(W')}(\text{ad}_{c^{-1}}(f_1)(\Omega(\partial_{f_2}^L \cdots \partial_{f_n}^L(x)))) \\ &\stackrel{(1.28)}{=} \text{ad}_c(f_1)(\mathcal{S}_{\mathcal{B}(W')}\Omega(\partial_{f_2}^L \cdots \partial_{f_n}^L(x))) \\ &= \text{ad}_c(f_1)(\cdots(\text{ad}_c(f_n)(\mathcal{S}_{\mathcal{B}(W')}\Omega(x)))) \\ &= \text{ad}_c(f_1)(\cdots(\text{ad}_c(f_n)(\mathcal{S}_{\mathcal{B}(W')}(x)))) \\ &= -\text{ad}_c(f_1)(\cdots(\text{ad}_c(f_n)(x))) \in L'_j. \end{aligned}$$

Since $\tilde{\Omega}(x) = -x$ for all $x \in L_j^{\max} = M'_{\beta_j}$, (3.32) and (3.33) imply that $\tilde{\Omega}(L_j) = L'_j$. We now prove that $\ker \tilde{\Omega} \cap L_j = \ker \Omega \cap L_j$ is a Yetter-Drinfeld module over $\mathcal{A}(V)$. Together with the irreducibility of L_j , see Prop. 3.5, this implies that $\tilde{\Omega}$ is injective and hence Claim (3.34) holds.

Since Ω is a map in ${}^H_H\mathcal{YD}$, see Prop. 3.14, one obtains that $\ker \Omega \cap L_j$ is an object in ${}^H_H\mathcal{YD}$. Further, for all $x \in L_j \cap \ker \Omega$ one has

$$\Omega(\mathcal{B}(V) \cdot x) = 0 \quad \text{by (3.31), and} \quad \Omega(V^* \cdot x) = 0 \quad \text{by (3.30).}$$

Thus $L_j \cap \ker \Omega$ is an object in ${}^{\mathcal{A}(V)}_{\mathcal{A}(V)}\mathcal{YD}$, and Claim (3.34) is proven.

Now we prove Theorem 3.12(2). Since Ω and $\mathcal{S}_{\mathcal{B}(W')}$ are \mathbb{Z}^θ -graded maps, (3.34) implies that $\text{supp } L_j = \text{supp } L'_j$ for all $j \in \mathbb{I} \setminus \{i\}$. In particular, $\text{supp } L'_j$ is finite for all $j \in \mathbb{I} \setminus \{i\}$, that is, Condition (F_i) is fulfilled for $M' = \mathcal{R}_i(M)$, and hence $M'' := \mathcal{R}_i(M')$ is well-defined. For all $j \in \mathbb{I}$ let $\gamma_j = s_{i,M'} s_{i,M}(\alpha_j)$. Then by Eq. (3.17) one obtains for all $j \in \mathbb{I} \setminus \{i\}$ the equations

$$(3.35) \quad -a_{ij}^{M'} = \sup\{h \in \mathbb{N}_0 \mid s_{i,M}(\alpha_j) + h s_{i,M}(\alpha_i) \in \text{supp } L'_j\} = -a_{ij}^M,$$

$$(3.36) \quad \gamma_j = s_{i,M}(\alpha_j) - a_{ij}^{M'} s_{i,M}(\alpha_i) = \alpha_j,$$

$$(3.37) \quad M''_j = L'_j \cap \mathcal{B}(W')_{\gamma_j} \simeq L_j \cap \mathcal{B}(W)_{\gamma_j} = M_j,$$

where the last equation follows from the fact that $\tilde{\Omega}|_{L_j} : L_j \rightarrow L'_j$ is a \mathbb{Z}^θ -graded isomorphism in ${}^H_H\mathcal{YD}$, see Prop. 3.14 and Claim (3.34). Since $M''_i = (M'_i)^* = M_i^{**} \simeq M_i$ by Remark 1.4, one obtains that $\mathcal{R}_i(M') \simeq M$, that is, Theorem 3.12(2) is proven.

It remains to prove that Ω is an isomorphism. Let $\mathcal{K}' = \mathcal{B}(W')^{\text{co } \mathcal{B}(M'_{\beta_i})}$, $W'' = \bigoplus_{j \in \mathbb{I}} M''_{\gamma_j}$, and $\mathcal{K}'' = \mathcal{B}(W'')^{\text{co } \mathcal{B}(M''_{\gamma_i})}$. Since \mathcal{K} resp. \mathcal{K}' is generated as an algebra by $\bigoplus_{j \in \mathbb{I} \setminus \{i\}} L_j$ resp. $\bigoplus_{j \in \mathbb{I} \setminus \{i\}} L'_j$, see Prop. 3.6, we conclude from Lemma 1.7 (ii) and Claim (3.34) that $\tilde{\Omega}(\mathcal{K}) = \mathcal{K}'$. By the same argument we have $\tilde{\Omega}'(\mathcal{K}') = \mathcal{K}''$, where $\tilde{\Omega}' : \mathcal{K}' \# \mathcal{B}(V^{**}) \rightarrow \mathcal{B}(W'')$ is the map in Prop. 3.14 obtained by starting with the family M' instead of M . Thus $\tilde{\Omega}$ and $\tilde{\Omega}'$ define surjective \mathbb{Z}^θ -homogeneous maps

$$\mathcal{K} \xrightarrow{\tilde{\Omega}|_{\mathcal{K}}} \mathcal{K}' \xrightarrow{\tilde{\Omega}'|_{\mathcal{K}'}} \mathcal{K}''$$

of Yetter-Drinfeld modules over H . But $\mathcal{K} \simeq \mathcal{K}''$ as \mathbb{Z}^θ -graded Yetter-Drinfeld modules since $W \simeq W''$ by Theorem 3.12 (2). The \mathbb{Z}^θ -homogeneous components of \mathcal{K} are all finite-dimensional since W is finite-dimensional. Hence the map $\mathcal{K} \xrightarrow{\tilde{\Omega}|_{\mathcal{K}}} \mathcal{K}' \xrightarrow{\tilde{\Omega}'|_{\mathcal{K}'}} \mathcal{K}''$ is bijective, and $\tilde{\Omega}|_{\mathcal{K}} : \mathcal{K} \rightarrow \mathcal{K}'$ is an

isomorphism. Next, let

$$\mu : \mathfrak{B}(V^*) \otimes \mathcal{K} \rightarrow \mathcal{K} \# \mathfrak{B}(V^*) \quad \text{and} \quad \mu' : \mathcal{K}' \otimes \mathfrak{B}(V^*) \rightarrow \mathfrak{B}(W')$$

be the multiplication maps. By Remark 2.7 resp. Lemma 3.2 (ii), both maps are bijective. Let $f \in \mathfrak{B}(V^*)$ and $x \in \mathcal{K}$. Then

$$\begin{aligned} \tilde{\Omega}(fx) &= \mathcal{S}_{\mathfrak{B}(W')}(f\Omega(x)) = (f_{(-1)} \cdot \tilde{\Omega}(x))\mathcal{S}_{\mathfrak{B}(W')}(f_{(0)}) \\ &= (f_{(-1)} \cdot \tilde{\Omega}(x))\mathcal{S}_{\mathfrak{B}(V^*)}(f_{(0)}). \end{aligned}$$

Thus $\tilde{\Omega}\mu = \mu'c(\mathcal{S}_{\mathfrak{B}(V^*)} \otimes (\mathcal{S}_{\mathfrak{B}(W')}\Omega|_{\mathcal{K}}))$. Hence $\tilde{\Omega}$, and *a fortiori* Ω , are bijective. This completes the proof of Theorem 3.12. \square

Remark 3.15. The proof of Theorem 3.12 does not use the fact that $V = M_i$ is irreducible in ${}^H_H\mathcal{YD}$. However, M_i has to be irreducible if one wants to apply the theorem for an index $j \in \mathbb{I}$, $j \neq i$, which satisfies Condition (F_j) .

The algebras $\mathcal{B}(W)$ and $\mathcal{B}(W')$ are not necessarily isomorphic. However, we have the following consequences of Theorem 3.12.

Corollary 3.16. *Let M , i , W , W' and the \mathbb{Z}^θ -gradings of W and W' be as in Thm. 3.12. Then $\mathcal{B}(W) \# \mathcal{B}(W^*)$ and $\mathcal{B}(W') \# \mathcal{B}(W'^*)$ are isomorphic as \mathbb{Z}^θ -graded objects in ${}^H_H\mathcal{YD}$. In particular, $\text{supp } \mathcal{B}(W) \# \mathcal{B}(W^*) = \text{supp } \mathcal{B}(W') \# \mathcal{B}(W'^*)$.*

Proof. Since the homogeneous components of \mathcal{K} are finite-dimensional, the graded dual $\mathcal{K}^{\text{gr-dual}}$ of \mathcal{K} is a \mathbb{Z}^θ -graded object in ${}^H_H\mathcal{YD}$. By definition of \mathcal{K} and the isomorphism $\mathcal{B}(W^*) \simeq \mathcal{B}(W)^{\text{gr-dual}}$, see (1.37), one has

$$\mathcal{B}(W) \# \mathcal{B}(W^*) \simeq \mathcal{K} \otimes \mathcal{B}(V) \otimes \mathcal{K}^{\text{gr-dual}} \otimes \mathcal{B}(V^*)$$

as \mathbb{Z}^θ -graded objects in ${}^H_H\mathcal{YD}$. Further, Theorem 3.12 implies that

$$\mathcal{B}(W') \# \mathcal{B}(W'^*) \simeq \mathcal{K} \otimes \mathcal{B}(V^*) \otimes \mathcal{K}^{\text{gr-dual}} \otimes \mathcal{B}(V)$$

as \mathbb{Z}^θ -graded objects in ${}^H_H\mathcal{YD}$. Since $A \otimes B \simeq B \otimes A$ for all \mathbb{Z}^θ -graded objects A, B in ${}^H_H\mathcal{YD}$, the above equations prove the corollary. \square

In many applications it will be more convenient to use the following reformulation of Corollary 3.16.

Corollary 3.17. *Let M , i , W , W' be as in Thm. 3.12. Define \mathbb{Z}^θ -gradings on W and W' by $\deg x = \alpha_j$ for all $x \in M_j$ and all $x \in M'_j$, $j \in \mathbb{I}$. Then for all $\alpha \in \mathbb{Z}^\theta$ the homogeneous components $(\mathcal{B}(W) \# \mathcal{B}(W^*))_\alpha$ and $(\mathcal{B}(W') \# \mathcal{B}(W'^*))_{s_{i,M}(\alpha)}$ are isomorphic in ${}^H_H\mathcal{YD}$. In particular,*

$$\text{supp } \mathcal{B}(W') \# \mathcal{B}(W'^*) = s_{i,M}(\text{supp } \mathcal{B}(W) \# \mathcal{B}(W^*)).$$

Corollary 3.18. *If $\dim \mathcal{B}(W) < \infty$, then $\dim \mathcal{B}(W) = \dim \mathcal{B}(W')$.*

Proof. We compute $\dim \mathcal{B}(W) = \dim \mathcal{K} \dim \mathcal{B}(V) = \dim \mathcal{K} \dim \mathcal{B}(V^*) = \dim \mathcal{B}(W')$. Here the first equality holds by Lemma 3.2 (ii), the second by Prop. 1.10, and the third by Theorem 3.12. \square

3.5. Weyl groupoid and real roots. In this subsection we define and study invariants of finite families of finite dimensional irreducible Yetter-Drinfeld modules. The definitions are based on Theorem 3.12.

Recall the definition of \mathcal{C}_θ from Subsection 3.4. If $M, M' \in \mathcal{C}_\theta$, then we say that

$$M \sim M'$$

if there exists an index i such that Condition (F_i) holds for M , see Subsection 3.4, and if $\mathcal{R}_i(M) \simeq M'$. By Theorem 3.12(2), the relation \sim is symmetric. The equivalence relation \approx generated by \sim is called *Weyl equivalence*.

Recall that $(\alpha_1, \dots, \alpha_\theta)$ is the standard basis of \mathbb{Z}^θ and $\mathbb{I} = \{1, 2, \dots, \theta\}$.

Definition 3.19. Let $M \in \mathcal{C}_\theta$. Define

$$\mathfrak{W}(M) = \{M' \in \mathcal{C}_\theta \mid M' \approx M\}.$$

Let $\mathcal{W}(M)$ denote the following category with $\text{Ob}(\mathcal{W}(M)) = \mathfrak{W}(M)$. For each $M' \in \mathfrak{W}(M)$ such that M' satisfies (F_i) consider the reflection $s_{i,M'} \in \text{Aut}(\mathbb{Z}^\theta)$, $s_{i,M'}(\alpha_j) = \alpha_j - a_{ij}^{M'} \alpha_i$ for all $j \in \mathbb{I}$, as a morphism $M' \rightarrow \mathcal{R}_i(M')$. Let $\mathcal{W}(M)$ be the category in which all morphisms are compositions of the morphisms $s_{i,M'}$, where $i \in \mathbb{I}$ and $M' \in \mathfrak{W}(M)$ satisfies (F_i) . The category $\mathcal{W}(M)$ is called the *Weyl groupoid of M* .

Let

$$\Delta^{re}(M) = \{w(\alpha_j) \mid w \in \text{Hom}(M', M), M' \in \mathfrak{W}(M)\} \subset \mathbb{Z}^\theta.$$

Following the notation in [K95, §5.1], $\Delta^{re}(M)$ is called the *set of real roots of M* .

Remark 3.20. Let $M \in \mathcal{C}_\theta$. Then the category $\mathcal{W}(M)$ is a connected groupoid. Indeed, if $i \in \mathbb{I}$ and $M' \in \mathfrak{W}(M)$ satisfies (F_i) , then $\mathcal{R}_i(M')$ satisfies (F_i) , $\mathcal{R}_i(\mathcal{R}_i(M')) \simeq M'$ and $s_{i,\mathcal{R}_i(M')} s_{i,M'} = \text{id}_{\mathbb{Z}^\theta}$ by Theorem 3.12. Therefore the generating morphisms (and hence all morphisms) of $\mathcal{W}(M)$ are invertible. Further, for any two $M', M'' \in \mathfrak{W}(M)$ there is a morphism in $\text{Hom}(M', M'')$ by the definition of $\mathfrak{W}(M)$.

Remark 3.21. If $M' \in \mathfrak{W}(M)$, then equation

$$\dim \mathcal{B}(\oplus_{n \in \mathbb{I}} M_n) = \dim \mathcal{B}(\oplus_{n \in \mathbb{I}} M'_n)$$

holds by Corollary 3.18.

Remark 3.22. Assume that M is of diagonal type, that is, $\dim M_n = 1$ for all $n \in \mathbb{I}$. Let $M' \in \mathfrak{W}(M)$, $j \in \mathbb{I}$, $k \in \mathbb{N}$, $i_1, \dots, i_k \in \mathbb{I}$, $M^0 = M'$, $M^1, M^2, \dots, M^k = M \in \mathfrak{W}(M)$, $M^l \simeq \mathcal{R}_{i_{l+1}}(M^{l+1})$ for all $l \in \mathbb{N}_0$ with $l < k$, and $\beta = s_{i_k, M^{k-1}} \cdots s_{i_2, M^1} s_{i_1, M^0}(\alpha_j) \in \Delta^{re}(M)$. Let $m_1, \dots, m_\theta \in \mathbb{Z}$ such that $\beta = \sum_{n \in \mathbb{I}} m_n \alpha_n$. Then $M'_j \simeq \otimes_{i \in \mathbb{I}} M_n^{\otimes m_n}$ in ${}^H_H \mathcal{YD}$, where $M_n^{\otimes m_n} = (M_n^*)^{\otimes -m_n}$ if $m_n < 0$. This means, that M' depends only on M and β , but not on the particular choice of i_1, \dots, i_k . However, for more general Yetter-Drinfeld modules M it is in general not clear, if M'_j can be recovered from j , β and M .

Proposition 3.23. *Let $M \in \mathcal{C}_\theta$ and $W = M_1 \oplus \cdots \oplus M_\theta$. Then $\Delta^{re}(M) \subset \text{supp } \mathcal{B}(W) \# \mathcal{B}(W^*)$. In particular, if $\mathcal{B}(W)$ is finite-dimensional, then $\Delta^{re}(M)$ is a finite subset of \mathbb{Z}^θ .*

Proof. Clearly, $\text{supp } W = \{\alpha_1, \dots, \alpha_\theta\} \subset \text{supp } \mathcal{B}(W) \# \mathcal{B}(W^*)$. Let $i \in \mathbb{I}$, $M' = \mathcal{R}_i(M)$, and $W' = \oplus_{n \in \mathbb{I}} M'_n$. Then

$$\text{supp } W' \subset \text{supp } \mathcal{B}(W') \# \mathcal{B}(W'^*) = s_{i,M}(\text{supp } \mathcal{B}(W) \# \mathcal{B}(W^*))$$

by Corollary 3.17. Thus equation $s_{i,M'} s_{i,M} = \text{id}$ gives that $s_{i,M'}(\alpha_j) \in \text{supp } \mathcal{B}(W) \# \mathcal{B}(W^*)$ for all $j \in \mathbb{I}$. By iteration one obtains that $\Delta^{re}(M) \subset \text{supp } \mathcal{B}(W) \# \mathcal{B}(W^*)$. If $\dim \mathcal{B}(W) < \infty$ then the finiteness of $\Delta^{re}(M)$ follows from the equations

$$\begin{aligned} \text{supp } \mathcal{B}(W) \otimes \mathcal{B}(W^*) &= \text{supp } \mathcal{B}(W) + \text{supp } \mathcal{B}(W)^{\text{gr-dual}} \\ &= \text{supp } \mathcal{B}(W) - \text{supp } \mathcal{B}(W), \end{aligned}$$

see (1.37), and the fact that $\text{supp } \mathcal{B}(W)$ is finite. \square

Lemma 3.24. *Let $M \in \mathcal{C}_\theta$ and let $i, j \in \mathbb{I}$ such that $a_{ij}^M = 0$. Then $a_{ji}^M = 0$, and $\mathcal{B}(M_i \oplus M_j) \simeq \mathcal{B}(M_i) \otimes \mathcal{B}(M_j)$ as graded vector spaces.*

Proof. Let $x \in M_i$, $y \in M_j$. Then (1.27) gives that

$$(3.38) \quad \Delta(\text{ad}_c x(y)) = \text{ad}_c x(y) \otimes 1 + x \otimes y - c^2(x \otimes y) + 1 \otimes \text{ad}_c x(y).$$

Thus, $a_{ij}^M = 0$ implies that $\text{ad}_c x(y) = 0$. Hence $x \otimes y - c^2(x \otimes y) = 0$, that is, $(\text{id} - c^2)(M_i \otimes M_j) = 0$. Then $(\text{id} - c^2)c(M_j \otimes M_i) = 0$, but c is invertible, so that $(\text{id} - c^2)(M_j \otimes M_i) = 0$. Eq. (3.38) gives that $\text{ad}_c x(y)$ is primitive in $\mathcal{B}(M_i \oplus M_j)$ for all $x \in M_j$, $y \in M_i$, hence zero. This yields $a_{ji}^M = 0$. The last claim of the lemma is [Gñ00, Thm. 2.2]. \square

Lemma 3.25. *Let $M = (M_j)_{j \in \mathbb{I}}$ be an object in \mathcal{C}_θ which satisfies (F_i) for all $i \in \mathbb{I}$. Then $A = (a_{ij}^M)$ is a generalized Cartan matrix. In particular, the subgroup*

$$\mathcal{W}_0(M) := \langle s_{i,M} \mid i \in \mathbb{I} \rangle$$

of $GL(\theta, \mathbb{Z})$ is isomorphic to the Weyl group of the Kac-Moody algebra $\mathfrak{g}(A)$.

Proof. The first claim follows from Lemma 3.24. Let $(\mathfrak{h}, \Pi, \Pi^\vee)$ be a realization of A [K95, §1.1] and let W be the Weyl group of $\mathfrak{g}(A)$ [K95, §3.7]. Then W preserves the subspace V of \mathfrak{h}^* generated by Π^\vee and the morphism $W \rightarrow GL(V)$ is injective [K95, Ex. 3.6]. Now $V \simeq \mathbb{Z}^\theta \otimes_{\mathbb{C}} \mathbb{C}$ by [K95, (1.1.1)] and the image of W in $GL(V)$ coincides with $\mathcal{W}_0(M)$ by [K95, (1.1.2)]. \square

It follows that $\mathcal{W}_0(M)$ is a Coxeter group [K95, Prop. 3.13] but we do not need this fact in the sequel. The group $\mathcal{W}_0(M)$ is important in the study of Nichols algebras in the following special case.

Definition 3.26. We say that $M \in \mathcal{C}_\theta$ is *standard* if M' satisfies Condition (F_i) and $a_{ij}^{M'} = a_{ij}^M$ for all $M' \in \mathfrak{W}(M)$ and $i, j \in \mathbb{I}$.

Remark 3.27. In the following two special cases the family $M \in \mathcal{C}_\theta$ is standard.

1. Let H be the group algebra of an abelian group Γ and M a family of 1-dimensional Yetter-Drinfeld modules $\mathbb{k}v_i = M_i$ over H , where $i \in \mathbb{I}$. Let $\delta(v_i) = g_i \otimes v_i$ and $g \cdot v_i = \chi_i(g)v_i$ denote the coaction and action of H , respectively, where $g_i \in \Gamma$, $\chi_i \in \widehat{\Gamma}$, $i \in \mathbb{I}$. Define $q_{ij} = \chi_j(g_i) \in \mathbb{k}$ for $i, j \in \mathbb{I}$. If M is of Cartan type, that is, for all $i \neq j$ there exist $a_{ij} \in \mathbb{Z}$ such that $0 \leq -a_{ij} < \text{ord } q_{ii}$ and $q_{ij}q_{ji} = q_{ii}^{a_{ij}}$, then M is standard. This can be seen from [H06, Lemma 1(ii), Eq. (24)].

2. Assume that $M \in \mathcal{C}_\theta$ satisfies Condition (F_i) and $\mathcal{R}_i(M)_{s_{i,M}(\alpha_j)} \simeq M_j$ in ${}^H_H\mathcal{YD}$ for all $i, j \in \mathbb{I}$. Then M is standard by definition of a_{ij}^M .

Lemma 3.28. *Assume that $M \in \mathcal{C}_\theta$ is standard. Then*

$$\Delta^{re}(M) = \{w(\alpha_j) \mid w \in \mathcal{W}_0(M), j \in \mathbb{I}\}.$$

In particular, $w(\Delta^{re}(M)) = \Delta^{re}(M)$ for all $w \in \mathcal{W}_0(M)$.

Proof. This follows immediately from the definitions of $\Delta^{re}(M)$ and $\mathcal{W}_0(M)$ and the relations $a_{ij}^{M'} = a_{ij}^M$ for all $i, j \in \mathbb{I}$ and $M' \in \mathfrak{W}(M)$. \square

Theorem 3.29. *Let $M = (M_i)_{i \in \mathbb{I}} \in \mathcal{C}_\theta$ and $W = \oplus_{i \in \mathbb{I}} M_i$. If M is standard and $\dim \mathcal{B}(W) < \infty$, then the generalized Cartan matrix $(a_{ij}^M)_{i,j \in \mathbb{I}}$ is of finite type.*

Proof. Since $\dim \mathcal{B}(W) < \infty$, the set $\Delta^{re}(M)$ is finite by Prop. 3.23. Since M is standard, $\Delta^{re}(M)$ is stable under the action of $\mathcal{W}_0(M)$ by Lemma 3.28. The corresponding permutation representation $\mathcal{W}_0(M) \rightarrow S(\Delta^{re}(M))$ is injective, since $\mathcal{W}_0(M) \subset GL(\theta, \mathbb{Z})$ and $\Delta^{re}(M)$ contains the standard basis of \mathbb{Z}^θ . Therefore $\mathcal{W}_0(M)$ is finite. Thus the claim follows from Lemma 3.25 and [K95, Prop. 4.9]. \square

4. APPLICATIONS

4.1. Hopf algebras with few finite-dimensional Nichols algebras.

Lemma 4.1. *Let H be a Hopf algebra. Assume that, up to isomorphism, there is exactly one finite-dimensional irreducible Yetter-Drinfeld module $L \in {}^H_H\mathcal{YD}$ such that $\dim \mathcal{B}(L) < \infty$. Let $M = (M_1, M_2) \in \mathcal{C}_2$ such that $M_1 \simeq M_2 \simeq L$.*

- (i) *If M satisfies (F_1) then M satisfies (F_2) and $a_{12}^M = a_{21}^M$. If additionally $\dim \mathcal{B}(L^2) < \infty$ then M is standard.*
- (ii) *If M does not fulfill (F_1) or if $a_{12}^M \leq -2$, then $\dim(\mathcal{B}(L^n)) = \infty$ for $n \geq 2$.*
- (iii) *If $a_{12}^M = 0$, then $\dim \mathcal{B}(L^n) = (\dim \mathcal{B}(L))^n$ for all $n \in \mathbb{N}$.*
- (iv) *When $a_{12}^M = -1$, then $\dim \mathcal{B}(L^n) = \infty$ for $n \geq 3$.*

Note that if $a_{12}^M = -1$ then Lemma 4.1 gives no information about $\dim \mathcal{B}(L^2)$.

Proof. If M does not fulfill Condition (F_1) then $\dim \mathcal{B}(L^2) = \infty$. Otherwise $a_{12}^M \in \mathbb{Z}_{\leq 0}$, and $a_{12}^M = a_{21}^M$ by symmetry. Moreover, if $\dim \mathcal{B}(L^2) < \infty$, then for $i \in \{1, 2\}$ the Nichols algebra of $\mathcal{R}_i(M)_1 \oplus \mathcal{R}_i(M)_2$ is also finite-dimensional by Corollary 3.18, and hence $\mathcal{R}_i(M)_j \simeq L$ for $j \in \{1, 2\}$. Therefore M is standard, and (i) is proven.

The generalized Cartan matrix $\begin{pmatrix} 2 & a_{12}^M \\ a_{12}^M & 2 \end{pmatrix}$ is of finite type iff $a_{12}^M = 0$ or $a_{12}^M = -1$. Then (ii) follows from Theorem 3.29. Now (iii) follows from [Gñ00], see Lemma 3.24. If $a_{12}^M = -1$, then the generalized Cartan matrix of L^3 has Dynkin diagram $A_2^{(1)}$; hence $\dim \mathcal{B}(L^3) = \infty$, and *a fortiori* the same holds for L^n for $n \geq 3$. This shows (iv). \square

Theorem 4.2. *Let H be a Hopf algebra such that the category of finite-dimensional Yetter-Drinfeld modules is semisimple. Assume that up to isomorphism there is exactly one irreducible $L \in {}^H_H\mathcal{YD}$ such that $\dim \mathcal{B}(L) < \infty$.*

Let $M = (M_1, M_2) \in \mathcal{C}_2$, where $M_1 = M_2 = L$. If M satisfies (F_1) then M satisfies (F_2) and $a_{12}^M = a_{21}^M \in \mathbb{Z}_{\leq 0}$.

- (i) If $a_{12}^M = -\infty$ or $a_{12}^M \leq -2$, then L is the only Yetter-Drinfeld module over H with finite-dimensional Nichols algebra.
- (ii) If $a_{12}^M = 0$, then a Yetter-Drinfeld module W over H has finite-dimensional Nichols algebra if and only if $W \simeq L^n$ for some $n \in \mathbb{N}$. Furthermore, $\dim \mathcal{B}(L^n) = (\dim \mathcal{B}(L))^n$.
- (iii) If $a_{12}^M = -1$, then the only possible Yetter-Drinfeld modules over H with finite-dimensional Nichols algebra are L and (perhaps) L^2 .

Proof. By hypothesis, the only Yetter-Drinfeld module candidates to have finite-dimensional Nichols algebras are those of the form L^n , $n \in \mathbb{N}$. The theorem follows then from Lemma 4.1. \square

Now we state another general result that can be obtained from Theorem 3.29. We shall use it when considering Nichols algebras over \mathbb{S}_4 .

Lemma 4.3. *Let $M_1, \dots, M_s \in {}^H_H\mathcal{YD}$, where $s \in \mathbb{N}$, be a maximal set of pairwise nonisomorphic irreducible Yetter-Drinfeld modules, such that $\dim \mathcal{B}(M_i) < \infty$ for $1 \leq i \leq s$. Assume that there exist $i, j \in \{1, \dots, s\}$ (the possibility $i = j$ is not excluded) such that*

- (i) $\dim \mathcal{B}(M_i \oplus M_j) < \infty$.
- (ii) If $\{\ell, m\} \neq \{i, j\}$, then $\dim \mathcal{B}(M_\ell \oplus M_m) = \infty$.
- (iii) $M_i \not\simeq M_j^*$.

Let $M = (M_i, M_j) \in \mathcal{C}_2$. Then M is standard.

Proof. By (i) the Nichols algebra of $(M_i \oplus M_j)^* \simeq M_i^* \oplus M_j^*$ is finite-dimensional. By (ii) one has $M_i^* \oplus M_j^* \simeq M_i \oplus M_j$, and (iii) implies that $M_i^* \simeq M_i$ and $M_j^* \simeq M_j$. Thus it suffices to consider the reflection \mathcal{R}_i . By (i) and Corollary 3.18, $M' = (M'_1, M'_2) := \mathcal{R}_i(M)$ is well-defined and $\dim \mathcal{B}(M'_1 \oplus M'_2) < \infty$. By (ii) one has $M'_1 \oplus M'_2 \simeq M_i \oplus M_j$. Since $M'_1 \simeq M_i^* \simeq M_i$ by the beginning of the proof, one has $M'_2 \simeq M_j$. Hence M is standard by Remark 3.27. \square

4.2. Pointed Hopf algebras with group \mathbb{S}_3 . In the rest of this section, it is assumed that the base field is $\mathbb{k} = \mathbb{C}$. Let G be a finite non-abelian group. We shall use the rack notation $x \triangleright y := xyx^{-1}$, $x, y \in G$. Since the group algebra $\mathbb{C}G$ is semisimple, the corresponding category ${}^{\mathbb{C}G}_G\mathcal{YD}$ of Yetter-Drinfeld modules is semisimple. It is well-known that the irreducible objects in ${}^{\mathbb{C}G}_G\mathcal{YD}$ are parametrized by pairs (\mathcal{O}, ρ) , where \mathcal{O} is a conjugacy class of G and ρ is an irreducible representation of the centralizer G^s of a fixed $s \in \mathcal{O}$.

Let $M(\mathcal{O}, \rho)$ denote the irreducible Yetter-Drinfeld module corresponding to (\mathcal{O}, ρ) and let $\mathcal{B}(\mathcal{O}, \rho)$ be its Nichols algebra. Then $M(\mathcal{O}, \rho)$ is the induced module $\text{Ind}_{G^s}^G \rho$, and the comodule structure is given by the following rule. Let $g_1 = g, \dots, g_t$ be a numeration of \mathcal{O} and let $x_i \in G$ such that $x_i \triangleright g = g_i$ for all $1 \leq i \leq t$. Then $M(\mathcal{O}, \rho) = \bigoplus_{1 \leq i \leq t} x_i \otimes V$. If $x_i v := x_i \otimes v \in M(\mathcal{O}, \rho)$, then $\delta(x_i v) = g_i \otimes x_i v$, for $1 \leq i \leq t, v \in V$. The braiding in $M(\mathcal{O}, \rho)$ is given by $c(x_i v \otimes x_j w) = g_i \cdot (x_j w) \otimes x_i v = x_h \rho(\gamma)(w) \otimes x_i v$ for any $1 \leq i, j \leq t, v, w \in V$, where $g_i x_j = x_h \gamma$ for unique $h, 1 \leq h \leq t$ and $\gamma \in G^s$.

If $G = \mathbb{S}_n$, then \mathcal{O}_2^n is the conjugacy class of the involutions and sgn is the restriction of the sign representation to the isotropy group.

Before stating our first classification result, we need to recall the construction of some Hopf algebras from [AG03a].

Definition 4.4. Let $\lambda \in \mathbb{k}$. Let $\mathcal{A}(\mathbb{S}_3, \mathcal{O}_2^3, \lambda)$ be the algebra presented by generators $e_t, t \in T := \{(12), (23)\}$, and $a_\sigma, \sigma \in \mathcal{O}_2^3$; with relations

$$(4.1) \quad e_t e_s e_t = e_s e_t e_s, \quad e_t^2 = 1, \quad s \neq t \in T;$$

$$(4.2) \quad e_t a_\sigma = -a_{t\sigma t} e_t \quad t \in T, \sigma \in \mathcal{O}_2^3;$$

$$(4.3) \quad a_\sigma^2 = 0, \quad \sigma \in \mathcal{O}_2^3;$$

$$(4.4) \quad a_{(12)} a_{(23)} + a_{(23)} a_{(13)} + a_{(13)} a_{(12)} = \lambda(1 - e_{(12)} e_{(23)});$$

$$(4.5) \quad a_{(12)} a_{(13)} + a_{(13)} a_{(23)} + a_{(23)} a_{(12)} = \lambda(1 - e_{(23)} e_{(12)}).$$

Set $e_{(13)} = e_{(12)} e_{(23)} e_{(12)}$. Then $\mathcal{A}(\mathbb{S}_3, \mathcal{O}_2^3, \lambda)$ is a Hopf algebra of dimension 72 with comultiplication determined by

$$(4.6) \quad \Delta(a_\sigma) = a_\sigma \otimes 1 + e_\sigma \otimes a_\sigma, \quad \Delta(e_t) = e_t \otimes e_t, \quad \sigma \in \mathcal{O}_2^3, t \in T.$$

Observe that the Hopf algebra $\mathcal{A}(\mathbb{S}_3, \mathcal{O}_2^3, \lambda)$ is isomorphic to $\mathcal{A}(\mathbb{S}_3, \mathcal{O}_2^3, \lambda c^2)$ (via $a_\sigma \mapsto c^{-1} a'_\sigma$, where a'_σ are the generators of the latter). Also $\mathcal{A}(\mathbb{S}_3, \mathcal{O}_2^3, 0) \simeq \mathcal{B}(\mathcal{O}_2^3, \text{sgn}) \# \mathbb{k}\mathbb{S}_3$. But $\mathcal{A}(\mathbb{S}_3, \mathcal{O}_2^3, 0) \not\simeq \mathcal{A}(\mathbb{S}_3, \mathcal{O}_2^3, 1)$ since the former is self-dual but the latter is not.

Theorem 4.5. *Let H be a finite-dimensional pointed Hopf algebra with $G(H) \simeq \mathbb{S}_3$. Then either $H \simeq \mathbb{k}\mathbb{S}_3$, or $H \simeq \mathcal{B}(\mathcal{O}_2^3, \text{sgn}) \# \mathbb{k}\mathbb{S}_3$ or $H \simeq \mathcal{A}(\mathbb{S}_3, \mathcal{O}_2^3, 1)$.*

Proof. It is known that $\dim \mathcal{B}(\mathcal{O}_2^3, \text{sgn}) = 12$ [MS00]; it is also known that this is the only finite-dimensional Nichols algebra with irreducible Yetter-Drinfeld module of primitives [AZ07]. We can then apply Theorem 4.2. Let $M = (M(\mathcal{O}_2^3, \text{sgn}), M(\mathcal{O}_2^3, \text{sgn}))$. Assume that $a_{12}^M \in \mathbb{Z}_{\leq 0}$, notation as above. We claim that $-a_{12}^M \geq 2$.

Let $\sigma_1 = (12)$, $\sigma_2 = (23)$, $\sigma_3 = (13) \in \mathbb{S}_3$. The Yetter-Drinfeld module $M(\mathcal{O}_2^3, \text{sgn}) \oplus M(\mathcal{O}_2^3, \text{sgn})$ has a basis x_1, x_2, x_3 (from the first copy), y_1, y_2, y_3 (from the second copy) with

$$(4.7) \quad \delta(x_i) = \sigma_i \otimes x_i, \delta(y_i) = \sigma_i \otimes y_i, t \cdot x_i = \text{sgn}(t)x_{t \triangleright i}, t \cdot y_i = \text{sgn}(t)y_{t \triangleright i},$$

for $1 \leq i \leq 3$, $t \in \mathbb{S}_3$. Here $\sigma_{t \triangleright i} := t \triangleright \sigma_i = t\sigma_i t^{-1}$. Also, $j \triangleright i$ means $\sigma_{j \triangleright i} := \sigma_j \triangleright \sigma_i$. The braiding in the vectors of the basis gives

$$\begin{aligned} c(x_j \otimes x_i) &= -x_{j \triangleright i} \otimes x_j, & c(y_j \otimes y_i) &= -y_{j \triangleright i} \otimes y_j, \\ c(x_j \otimes y_i) &= -y_{j \triangleright i} \otimes x_j, & c(y_j \otimes x_i) &= -x_{j \triangleright i} \otimes y_j. \end{aligned}$$

To prove our claim, we need to find i, j, k such that $\text{ad}_c(x_i)(\text{ad}_c(x_j)(y_k)) \neq 0$. Let $\partial_{x_i}, \partial_{y_i}$ be the skew-derivations as in [MS00]. Now

$$\begin{aligned} \text{ad}_c(x_2)(\text{ad}_c(x_1)(y_2)) &= \text{ad}_c(x_2)(x_1 y_2 + y_3 x_1) \\ &= x_2 x_1 y_2 + x_2 y_3 x_1 - x_3 y_2 x_2 - y_1 x_3 x_2, \end{aligned}$$

hence $\partial_{x_3} \partial_{y_1}(\text{ad}_c(x_2)(\text{ad}_c(x_1)(y_2))) = \partial_{x_3}(-x_2 x_3) = -x_2 \neq 0$, and the claim is proved. Thus $\dim \mathcal{B}(M(\mathcal{O}_2^3, \text{sgn}) \oplus M(\mathcal{O}_2^3, \text{sgn})) = \infty$ by Theorem 4.2, and $\mathcal{B}(\mathcal{O}_2^3, \text{sgn})$ is the only finite-dimensional Nichols algebra over \mathbb{S}_3 .

Let $H \not\simeq \mathbb{k}\mathbb{S}_3$ be a finite-dimensional pointed Hopf algebra with $G(H) \simeq \mathbb{S}_3$. Then the infinitesimal braiding of H , see [AS02], is isomorphic to $M(\mathcal{O}_2^3, \text{sgn})$. Hence H is generated as algebra by group-like and skew-primitive elements [AG03a, Theorem 2.1] and the theorem follows from [AG03a, Thm. 3.8]. \square

4.3. Nichols algebras over the group \mathbb{S}_4 . Let us recall the general terminology for \mathbb{S}_n . If $\pi = (12) \in \mathcal{O}_2^n$, then the isotropy subgroup is $\mathbb{S}_n^\pi \simeq \mathbb{Z}_2 \times \mathbb{S}_{n-2}$. Any irreducible representation of \mathbb{S}_n^π is of the form $\chi \otimes \rho$, where $\chi \in \widehat{\mathbb{Z}_2}$, $\rho \in \widehat{\mathbb{S}_{n-2}}$. If $\chi = \varepsilon$, then $\chi \otimes \rho(\pi) = 1$ and $\dim \mathcal{B}(\mathcal{O}_2^n, \varepsilon \otimes \rho) = \infty$. Thus, we are actually interested in the Nichols algebras $\mathcal{B}(\mathcal{O}_2^n, \text{sgn} \otimes \rho)$. If $\rho = \text{sgn}$, then $\text{sgn} \otimes \rho$ is just the restriction to \mathbb{S}_n^π of the sign representation of \mathbb{S}_n ; we denote in this case $\mathcal{B}(\mathcal{O}_2^n, \text{sgn}) = \mathcal{B}(\mathcal{O}_2^n, \text{sgn} \otimes \text{sgn})$.

The proof of Theorem 4.5 gives the following result.

Lemma 4.6. *The Nichols algebras $\mathcal{B}(M(\mathcal{O}_2^n, \text{sgn} \otimes \rho) \oplus M(\mathcal{O}_2^n, \text{sgn} \otimes \rho'))$, $n \geq 4$, $\rho, \rho' \in \widehat{\mathbb{S}_{n-2}}$, have infinite dimension.*

Proof. The braided vector space $M(\mathcal{O}_2^3, \text{sgn}) \oplus M(\mathcal{O}_2^3, \text{sgn})$ is a braided subspace of any of these braided vector spaces. \square

The isotropy group of the 4-cycle (1234) in \mathbb{S}_4 is the cyclic group $\langle (1234) \rangle$. Let χ_- be its character defined by $\chi_-(1234) = -1$. Let \mathcal{O}_4^4 be the conjugacy class of 4-cycles in \mathbb{S}_4 .

Theorem 4.7. *The only Nichols algebras of Yetter-Drinfeld modules over \mathbb{S}_4 with finite dimension, up to isomorphism, are those in the following list. All of them have dimension 576.*

- (1) $\mathcal{B}(\mathcal{O}_2^4, \text{sgn})$.
- (2) $\mathcal{B}(\mathcal{O}_2^4, \text{sgn} \otimes \varepsilon)$.
- (3) $\mathcal{B}(\mathcal{O}_4^4, \chi_-)$.

Proof. The Nichols algebras in the list have the claimed dimension by [FK99, MS00, AG03b], respectively. These are the only Nichols algebras of irreducible Yetter-Drinfeld modules over \mathbb{S}_4 with finite dimension by [AZ07].

It remains to show: If M, M' are two of $M(\mathcal{O}_2^4, \text{sgn})$, $M(\mathcal{O}_2^4, \text{sgn} \otimes \varepsilon)$, $M(\mathcal{O}_4^4, \chi_-)$, then $\dim \mathcal{B}(M \oplus M') = \infty$. Some possibilities are covered by Lemma 4.6. The rest are:

- (i) $\mathcal{B}(M(\mathcal{O}_4^4, \chi_-) \oplus M(\mathcal{O}_4^4, \chi_-))$.
- (ii) $\mathcal{B}(M(\mathcal{O}_2^4, \text{sgn}) \oplus M(\mathcal{O}_4^4, \chi_-))$.
- (iii) $\mathcal{B}(M(\mathcal{O}_2^4, \text{sgn} \otimes \varepsilon) \oplus M(\mathcal{O}_4^4, \chi_-))$.

(i). We claim that there is a surjective rack homomorphism $\mathcal{O}_4^4 \rightarrow \mathcal{O}_2^3$ that induces a surjective morphism of braided vector spaces $M(\mathcal{O}_4^4, \chi_-) \oplus M(\mathcal{O}_4^4, \chi_-) \rightarrow M(\mathcal{O}_2^3, \text{sgn}) \oplus M(\mathcal{O}_2^3, \text{sgn})$; since the Nichols algebra of the latter is infinite-dimensional by the proof of Theorem 4.5, $\dim \mathcal{B}(M(\mathcal{O}_4^4, \chi_-) \oplus M(\mathcal{O}_4^4, \chi_-)) = \infty$ too. Let us now verify the claim. We numerate the elements in the orbit \mathcal{O}_4^4 as follows:

$$\begin{aligned} \tau_1 &= (1234), & \tau_3 &= (1243), & \tau_5 &= (1324), \\ \tau_2 &= (1432) = \tau_1^{-1}, & \tau_4 &= (1342) = \tau_3^{-1}, & \tau_6 &= (1423) = \tau_5^{-1}; \end{aligned}$$

set accordingly

$$h_1 = \tau_1, \quad h_2 = (24), \quad h_3 = \tau_6, \quad h_4 = \tau_5, \quad h_5 = \tau_3, \quad h_6 = \tau_4;$$

so that $h_i \triangleright \tau_1 = \tau_i$, $1 \leq i \leq 6$. The Yetter-Drinfeld module $M(\mathcal{O}_4^4, \chi_-) \oplus M(\mathcal{O}_4^4, \chi_-)$ has a basis u_1, \dots, u_6 (from the first copy), w_1, \dots, w_6 (from the second copy) with

$$(4.8) \quad \begin{aligned} \delta(u_i) &= \tau_i \otimes u_i, & t \cdot u_i &= \chi_-(\tilde{t})u_{t \triangleright i}, \\ \delta(w_i) &= \tau_i \otimes w_i, & t \cdot w_i &= \chi_-(\tilde{t})w_{t \triangleright i}, \end{aligned}$$

for $1 \leq i \leq 6$, $t \in \mathbb{S}_4$. Here $t \triangleright i$ and $\tilde{t} \in \mathbb{S}_4^{\tau_1} = \langle \tau_1 \rangle$ have the meaning that $th_i = h_{t \triangleright i} \tilde{t}$. Let now

$$(4.9) \quad I_3 = \{1, 2\}, \quad I_2 = \{3, 4\}, \quad I_1 = \{5, 6\}.$$

Let $a, b \in \{1, 2, 3\}$, $i \in I_a$, $j \in I_b$. If $a = b$, then the braiding in the corresponding vectors of the basis is

$$\begin{aligned} c(u_i \otimes u_j) &= -u_j \otimes u_i, & c(u_i \otimes w_j) &= -w_j \otimes u_i, \\ c(w_i \otimes w_j) &= -w_j \otimes w_i, & c(w_i \otimes u_j) &= -u_j \otimes w_i; \end{aligned}$$

and if $a \neq b$, then for some $\ell \in I_c$, where $c \neq a, b$, one has

$$\begin{aligned} c(u_i \otimes u_j) &= -u_\ell \otimes u_i, & c(u_i \otimes w_j) &= -w_\ell \otimes u_i, \\ c(w_i \otimes w_j) &= -w_\ell \otimes w_i, & c(w_i \otimes u_j) &= -u_\ell \otimes w_i. \end{aligned}$$

Thus, the map $\pi : M(\mathcal{O}_4^4, \chi_-) \oplus M(\mathcal{O}_4^4, \chi_-) \rightarrow M(\mathcal{O}_2^3, \text{sgn}) \oplus M(\mathcal{O}_2^3, \text{sgn})$ given by $\pi(u_i) = x_a$, $\pi(w_i) = y_a$, for $i \in I_a$, $a = 1, 2, 3$, preserves the braiding. This proves the claim.

(ii). We claim that there is a surjective morphism of braided vector spaces $M(\mathcal{O}_2^4, \text{sgn}) \oplus M(\mathcal{O}_4^4, \chi_-) \rightarrow M(\mathcal{O}_2^3, \text{sgn}) \oplus M(\mathcal{O}_2^3, \text{sgn})$. Again, this implies that the Nichols algebra in (ii) is infinite-dimensional. Let us check the claim. We numerate the elements in the orbit \mathcal{O}_2^4 as follows:

$$\sigma_1 = (12), \quad \sigma_2 = (23), \quad \sigma_3 = (13), \quad \sigma_4 = (14), \quad \sigma_5 = (24), \quad \sigma_6 = (34);$$

set accordingly

$$g_1 = \sigma_1, \quad g_2 = \sigma_3, \quad g_3 = \sigma_2, \quad g_4 = \sigma_5, \quad g_5 = \sigma_4, \quad g_6 = (1324);$$

so that $g_i \triangleright \sigma_1 = \sigma_i$, $1 \leq i \leq 6$. Let τ_i and h_i , $1 \leq i \leq 6$, be as in the previous part of the proof. The Yetter-Drinfeld module $M(\mathcal{O}_4^4, \text{sgn}) \oplus M(\mathcal{O}_4^4, \chi_-)$ has a basis z_1, \dots, z_6 (from the first summand), w_1, \dots, w_6 (from the second summand) with

$$(4.10) \quad \begin{aligned} \delta(z_i) &= \sigma_i \otimes z_i, & t \cdot z_i &= \text{sgn}(t') z_{t \triangleright i}, \\ \delta(w_i) &= \tau_i \otimes w_i, & t \cdot w_i &= \chi_-(\tilde{t}) w_{t \triangleright i}, \end{aligned}$$

for $1 \leq i \leq 6$, $t \in \mathbb{S}_4$. Here, in the first line $t \triangleright i$ and $t' \in \mathbb{S}_4^{\sigma_1}$ have the meaning that $tg_i = g_{t \triangleright i} t'$; and in the second line, $t \triangleright i$ and $\tilde{t} \in \mathbb{S}_4^{\tau_1} = \langle \tau_1 \rangle$ have the meaning that $th_i = h_{t \triangleright i} \tilde{t}$. Set $t_1 := \sigma_1$, $t_2 := \sigma_6$, so that $\mathbb{S}_4^{\sigma_1} = \langle t_1, t_2 \rangle$.

Let now I_a be as in (4.9) and let $J_1 = \{1, 6\}$, $J_2 = \{2, 4\}$, $J_3 = \{3, 5\}$. Let $a, b, c \in \{1, 2, 3\}$ such that $\sigma_a \triangleright \sigma_b = \sigma_c$. Let $i \in I_a$, $j \in I_b$. Then there exist $k \in I_c$, $\ell, m \in J_c$, $\epsilon \in \{\pm 1\}$, $p, q \in \{1, 2\}$ such that

$$\sigma_i h_j = h_k \tau_1^\epsilon, \quad \sigma_i g_j = g_\ell t_p, \quad \tau_i g_j = g_m t_q;$$

see the Table 1.

TABLE 1. Multiplication in \mathbb{S}_4 .

\cdot	h_1	h_2	h_3	h_4	h_5	h_6
σ_1	$h_4\tau_1^{-1}$	$h_3\tau_1^{-1}$	$h_2\tau_1$	$h_1\tau_1$	$h_6\tau_1$	$h_5\tau_1^{-1}$
σ_2	$h_5\tau_1^{-1}$	$h_6\tau_1$	$h_4\tau_1$	$h_3\tau_1^{-1}$	$h_1\tau_1$	$h_2\tau_1^{-1}$
σ_3	$h_2\tau_1$	$h_1\tau_1$	$h_5\tau_1^{-1}$	$h_6\tau_1$	$h_3\tau_1^{-1}$	$h_4\tau_1^{-1}$
σ_4	$h_6\tau_1^{-1}$	$h_5\tau_1$	$h_4\tau_1^{-1}$	$h_3\tau_1$	$h_2\tau_1^{-1}$	$h_1\tau_1$
σ_5	$h_2\tau_1$	$h_1\tau_1^{-1}$	$h_6\tau_1^{-1}$	$h_5\tau_1^{-1}$	$h_4\tau_1$	$h_3\tau_1$
σ_6	$h_3\tau_1^{-1}$	$h_4\tau_1^{-1}$	$h_1\tau_1$	$h_2\tau_1$	$h_6\tau_1^{-1}$	$h_5\tau_1$

\cdot	g_1	g_2	g_3	g_4	g_5	g_6
σ_1	g_1t_1	g_3t_1	g_2t_1	g_5t_1	g_4t_1	g_6t_2
σ_2	g_3t_1	g_2t_1	g_1t_1	g_4t_2	g_6t_1	g_5t_1
σ_3	g_2t_1	g_1t_1	g_3t_1	g_6t_2	g_5t_2	g_4t_2
σ_4	g_5t_1	g_2t_2	g_6t_1	g_4t_1	g_1t_1	g_3t_1
σ_5	g_4t_1	g_6t_2	g_3t_2	g_1t_1	g_5t_1	g_2t_2
σ_6	g_1t_2	g_5t_2	g_4t_2	g_3t_2	g_2t_2	g_6t_1

\cdot	g_1	g_2	g_3	g_4	g_5	g_6
τ_1	g_2t_2	g_6t_1	g_5t_1	g_1t_2	g_3t_2	g_4t_1
τ_2	g_4t_2	g_1t_2	g_5t_2	g_6t_1	g_3t_1	g_2t_1
τ_3	g_5t_2	g_4t_2	g_1t_2	g_2t_1	g_6t_2	g_3t_2
τ_4	g_3t_2	g_4t_1	g_6t_2	g_2t_2	g_1t_2	g_5t_2
τ_5	g_6t_1	g_5t_1	g_2t_2	g_3t_1	g_4t_2	g_1t_2
τ_6	g_6t_2	g_3t_2	g_4t_1	g_5t_2	g_2t_1	g_1t_1

Hence, the braiding in the vectors of the basis is

$$c(z_i \otimes w_j) = -w_k \otimes z_i, \quad c(z_i \otimes z_j) = -z_\ell \otimes z_i, \quad c(w_i \otimes z_j) = -z_m \otimes w_i.$$

Thus, the map $\pi : M(\mathcal{O}_2^4, \text{sgn}) \oplus M(\mathcal{O}_4^4, \chi_-) \rightarrow M(\mathcal{O}_2^3, \text{sgn}) \oplus M(\mathcal{O}_2^3, \text{sgn})$ given by $\pi(z_i) = x_a$, $\pi(w_j) = y_a$, for $i \in I_a$, $j \in J_a$, $a = 1, 2, 3$, preserves the braiding. This proves the claim.

(iii). The argument in the preceding part can not be adapted to this one. However, assume that $\dim \mathcal{B}(M(\mathcal{O}_2^4, \text{sgn} \otimes \varepsilon) \oplus M(\mathcal{O}_4^4, \chi_-)) < \infty$. Then $M(\mathcal{O}_2^4, \text{sgn} \otimes \varepsilon) \oplus M(\mathcal{O}_4^4, \chi_-)$ is standard with finite Cartan matrix (a_{ij}) , by Lemma 4.3. Let σ_i and g_i , τ_i and h_i , $1 \leq i \leq 6$, be as in previous part of the proof. The Yetter-Drinfeld module $M(\mathcal{O}_4^4, \text{sgn} \otimes \varepsilon) \oplus M(\mathcal{O}_4^4, \chi_-)$ has a basis $\tilde{z}_1, \dots, \tilde{z}_6$ (from the first summand), w_1, \dots, w_6 (from the second summand)

with action and coaction given by $\delta(\tilde{z}_i) = \sigma_i \otimes \tilde{z}_i$, $t \cdot \tilde{z}_i = (\text{sgn} \otimes \varepsilon)(t') \tilde{z}_{t \triangleright i}$ for $1 \leq i \leq 6$, $t \in \mathbb{S}_4$, and the second line of (4.10). Here, $t \triangleright i$ and $t' \in \mathbb{S}_4^{\sigma_1}$ have the meaning that $tg_i = g_{t \triangleright i} t'$. Then

$$\begin{aligned} \text{ad}(\tilde{z}_2)(\text{ad}(\tilde{z}_1)(w_1)) &= \tilde{z}_2 \tilde{z}_1 w_1 + \tilde{z}_2 w_4 \tilde{z}_1 - \tilde{z}_3 w_5 \tilde{z}_2 - w_3 \tilde{z}_3 \tilde{z}_2 \neq 0 \\ \text{since } \partial_{\tilde{z}_1} \partial_{w_1}(\text{ad}(\tilde{z}_2)(\text{ad}(\tilde{z}_1)(w_1))) &= \partial_{\tilde{z}_1}(\tilde{z}_2 \tilde{z}_1) = \tilde{z}_2 \neq 0; \\ \text{ad}(w_2)(\text{ad}(w_1)(\tilde{z}_1)) &= w_2 w_1 \tilde{z}_1 - w_2 \tilde{z}_2 w_1 + w_1 \tilde{z}_4 w_2 - \tilde{z}_1 w_1 w_2 \neq 0 \\ \text{since } \partial_{w_5} \partial_{\tilde{z}_2}(\text{ad}(w_2)(\text{ad}(w_1)(\tilde{z}_1))) &= \partial_{w_5}(w_2 w_5) = w_2 \neq 0. \end{aligned}$$

Hence $a_{12} \leq -2$, $a_{21} \leq -2$, a contradiction. Thus, $\dim \mathcal{B}(M(\mathcal{O}_2^4, \text{sgn} \otimes \varepsilon) \oplus M(\mathcal{O}_4^4, \chi_-)) = \infty$. \square

4.4. Nichols algebras over the group \mathbb{D}_n , n odd. Let $n > 1$ be an odd integer and let \mathbb{D}_n be the dihedral group of order $2n$, generated by x and y with defining relations $x^2 = e = y^n$ and $xyx = y^{-1}$. Let \mathcal{O} be a conjugacy class of \mathbb{D}_n and let ρ be an irreducible representation of the centralizer G^s of a fixed $s \in \mathcal{O}$.

By [AF07, Th. 3.1], we know that there is at most one pair (\mathcal{O}, ρ) such that the Nichols algebra $\mathcal{B}(\mathcal{O}, \rho)$ is finite-dimensional, namely $(\mathcal{O}, \rho) = (\mathcal{O}_x, \text{sgn})$, where $\text{sgn} \in \widehat{\mathbb{D}_n^x}$, $\mathbb{D}_n^x = \langle x \rangle \simeq \mathbb{Z}_2$. However, it is not known if the dimension of $\mathcal{B}(\mathcal{O}_x, \text{sgn})$ is finite, except when $n = 3$ —since $\mathbb{D}_3 \simeq \mathbb{S}_3$.

The next result generalizes the first part of the proof of Theorem 4.5.

Theorem 4.8. *The only possible Nichols algebra over \mathbb{D}_n with finite dimension, up to isomorphism, is $\mathcal{B}(\mathcal{O}_x, \text{sgn})$.*

Proof. If $\dim \mathcal{B}(\mathcal{O}_x, \text{sgn}) = \infty$, then there is no finite-dimensional Nichols algebra over \mathbb{D}_n . Otherwise, we can apply Theorem 4.2. Let $\mathbb{M} = M(\mathcal{O}_x, \text{sgn}) \oplus M(\mathcal{O}_x, \text{sgn})$. Assume that $a_{12} \in \mathbb{Z}_{\leq 0}$, notation as above. We claim that $-a_{12} \geq 2$. Let $\sigma_i = xy^i \in \mathbb{D}_n$; $\mathcal{O}_x = \{\sigma_i \mid i \in \mathbb{Z}_n\}$. The Yetter-Drinfeld module \mathbb{M} has a basis v_i , $i \in \mathbb{Z}_n$ (from the first copy), w_i , $i \in \mathbb{Z}_n$ (from the second copy) with action, coaction and braiding

$$\begin{aligned} t \cdot v_i &= \text{sgn}(t) v_{t \triangleright i}, & t \cdot w_i &= \text{sgn}(t) w_{t \triangleright i}, \\ \delta(v_i) &= \sigma_i \otimes v_i, & \delta(w_i) &= \sigma_i \otimes w_i, \\ c(v_j \otimes v_i) &= -v_{j \triangleright i} \otimes v_j, & c(w_j \otimes w_i) &= -w_{j \triangleright i} \otimes w_j, \\ c(v_j \otimes w_i) &= -w_{j \triangleright i} \otimes v_j, & c(w_j \otimes v_i) &= -v_{j \triangleright i} \otimes w_j. \end{aligned}$$

for $i, j \in \mathbb{Z}_n$, $t \in \mathbb{D}_n$. Here, as above, $\sigma_{t \triangleright i} := t \triangleright \sigma_i = t \sigma_i t^{-1}$. To prove our claim, we need to find i, j, k such that $\text{ad}_c(v_i) \text{ad}_c(v_j)(w_k) \neq 0$. Let ∂_{v_i} , ∂_{w_i}

be the skew-derivations as in [MS00]. Now

$$\begin{aligned}\mathrm{ad}_c(v_2) \mathrm{ad}_c(v_1)(w_2) &= \mathrm{ad}_c(v_2)(v_1 w_2 + w_0 v_1) \\ &= v_2 v_1 w_2 + v_2 w_0 v_1 - v_3 w_2 v_2 - w_4 v_3 v_2,\end{aligned}$$

hence $\partial_{v_6} \partial_{w_4} (\mathrm{ad}_c(v_2) \mathrm{ad}_c(v_1)(w_2)) = \partial_{v_6} (-v_5 v_6) = -v_5 \neq 0$. The claim and the theorem are proved. \square

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